Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 57 (2007) 1193–1213
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

A lattice gauge model of singular Marsden–Weinstein reduction Part I. Kinematics

E. Fischer[a], G. Rudolph[b], M. Schmidt[b,*]

[a] *Max Planck Institute for the Physics of Complex Systems, Nöthnitzer Straße 38, 01187 Dresden, Germany*
[b] *Institute for Theoretical Physics, University of Leipzig, Augustusplatz 10/11, 04109 Leipzig, Germany*

Received 7 June 2006; received in revised form 8 September 2006; accepted 27 September 2006
Available online 3 November 2006

Abstract

The simplest nontrivial toy model of a classical SU(3) lattice gauge theory is studied in the Hamiltonian approach. By means of singular symplectic reduction, the reduced phase space is constructed. Two equivalent descriptions of this space in terms of a symplectic covering as well as in terms of invariants are derived.

MSC: 37J15; 70S15

Subject classification: Symplectic geometry; Classical mechanics

Keywords: Reduced phase space; Cotangent bundle reduction; Singular reduction; Stratified symplectic space; Invariant theory

1. Introduction

In the study of quantum gauge theory by nonperturbative methods there exist, in effect, two approaches: one is quantizing the unreduced system and then reducing the symmetries on the quantum level; the other one is first reducing the symmetries on the classical level and then quantizing the reduced system. For a discussion of the first strategy within the framework of lattice gauge theory, see [10,11] and the references therein. The aim of the present paper is to contribute to the second approach. The motivation behind stems from the well-known fact that nonabelian gauge fields can have several symmetry types, which give rise to singularities in the 'true' (i.e., reduced) classical configuration space. Speaking mathematically, the latter is a stratified space rather than a smooth manifold. It is natural to ask whether the singularities produce physical effects. For a systematic study of this open problem one needs a concept of how to implement the singularity structure in quantum theory. Such concepts have been developed in recent years; see, e.g., [6,7,12]. To separate the problem of symmetry reduction from the usual problems of a field theory related to the infinite number of degrees of freedom, it is reasonable to first study lattice gauge theory. In this way, one obtains a variety of toy models for forming and testing concepts and for developing quantum theory on a

* Corresponding author. Tel.: +49 341 9732431; fax: +49 341 9732548.
E-mail address: matthias.schmidt@itp.uni-leipzig.de (M. Schmidt).

0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2006.09.008

space with singularities. It is important for quantum theory, as well as interesting in its own right, to understand the classical dynamics of these models. Thus, in the present paper we consider the simplest nontrivial model of an SU(3) lattice gauge theory, where the lattice consists of a single plaquette. We study the kinematics of this model, i.e., the structure of the reduced phase space. The classical dynamics will then be studied in a subsequent paper.

We proceed as follows. In Section 2 we introduce the model. In Section 3 we carry out symmetry reduction. This will lead us to the so-called reduced cotangent bundle [13]. Then we give two equivalent descriptions of this bundle. One is in terms of a symplectic covering (Section 4), the other one is in terms of invariants (Section 5). We conclude with some general remarks on the dynamics in Section 6.

2. The model

Let us consider chromodynamics on a finite regular cubic lattice Λ. Denote the set of the i-dimensional elements of Λ by Λ^i (sites, links, plaquettes and cubes in increasing order of i). For neighbouring sites x, y, let (x, y) denote the link with orientation from x to y. Assume that we have chosen *one* orientation for each link. This means in particular that if (x, y) belongs to Λ^1 then (y, x) does not. For the effect of a change of the chosen link orientation on the description of gauge fields see Remark 2.1.

The gauge group is $G = \mathrm{SU}(3)$, its Lie algebra is $\mathfrak{g} = \mathrm{su}(3)$. The classical gluonic potential is approximated by its parallel transporter:

$$\Lambda^1 \ni (x, y) \mapsto a_{(x,y)} \in G.$$

Thus, the unreduced classical configuration space is the direct product G^{Λ^1} and the unreduced phase space is $\mathrm{T}^*G^{\Lambda^1}$. By means of the natural isomorphisms $\mathrm{T}^*G^{\Lambda^1} \cong (\mathrm{T}^*G)^{\Lambda^1}$, $\mathrm{T}^*G \cong G \times \mathfrak{g}^*$ and $\mathfrak{g}^* \cong \mathfrak{g}$, see below for details, the canonically conjugate momenta (colour electric fields) are given by maps

$$\Lambda^1 \ni (x, y) \mapsto A_{(x,y)} \in \mathfrak{g}.$$

Local gauge transformations are approximated by maps

$$\Lambda^0 \ni x \mapsto g_x \in G,$$

and hence the group of local gauge transformations is the direct product G^{Λ^0}. It acts on the phase space as follows:

$$a'_{(x,y)} = g_x \cdot a_{(x,y)} \cdot g_y^{-1}, \qquad A'_{(x,y)} = g_x \cdot A_{(x,y)} \cdot g_x^{-1}.$$

Remark 2.1. When reversing the orientation, the parallel transporter a is inverted and the associated momentum A, roughly, gets a minus sign. However, for A the situation is actually more delicate, as it is 'sitting' on the starting site of the link. This is a remnant of the approximation of classical fields we have chosen here. Since this is not relevant for the rest of the paper, for details we refer the reader to [11].

The (gauge invariant) Hamiltonian is given by

$$H = -\frac{\delta^3}{2} \sum_{(x,y)\in\Lambda^1} \mathrm{tr}(A^2_{(x,y)}) + \frac{1}{2g^2\delta} \sum_{p\in\Lambda^2} (6 - \mathrm{tr}(a_p + a_p^\dagger)). \tag{1}$$

Here, δ and g denote the lattice spacing and the coupling constant, respectively, and a_p is the parallel transporter around the plaquette p. For a plaquette p with vertices x, y, z, u we choose

$$a_p = a_{xy} \cdot a_{yz} \cdot a_{zu} \cdot a_{ux}.$$

While a_p depends on the choice of a base point x, $\mathrm{tr}(a_p)$ does not.

In the present paper we consider the case where Λ consists of a single plaquette. This is the simplest nontrivial model for a Hamiltonian lattice gauge theory. On three of the links of the plaquette, a and A can be gauged to $\mathbb{1}$ and 0, respectively. Such a gauge is called a tree gauge. Then the residual gauge freedom consists of constant gauge transformations. Thus, the unreduced configuration space is the group manifold G and the unreduced phase space is

$\mathrm{T}^*G \cong G \times \mathfrak{g}$. Its elements will be denoted by (a, A). The gauge group is G; its action on the phase space is given by diagonal conjugation

$$a' = gag^{-1}, \qquad A' = gAg^{-1}.$$

The Hamiltonian becomes

$$H = -\frac{\delta^3}{2}\mathrm{tr}(A^2) + \frac{1}{2g^2\delta}(6 - \mathrm{tr}(a + a^\dagger)). \tag{2}$$

Next, we will carry out symmetry reduction. The basic object for this is the G-manifold of the unreduced configuration space, because it determines the kinematical structure of the model completely.

3. Symmetry reduction

First, let us recall the general procedure. It is known as cotangent bundle reduction and is a special case of (singular) Marsden–Weinstein reduction.

3.1. Cotangent bundle reduction

Let Q be a manifold acted upon properly by a Lie group K (we may even assume that K is compact). Let $\mathfrak{k}$ denote the Lie algebra of K. Associated with (Q, K) there is the surjection

$$\pi : \mathrm{T}^*(Q/K) \to Q/K. \tag{3}$$

The base space Q/K consists of the K-orbits in Q, equipped with the quotient topology, the stratification by the orbit types of K-action and the smooth structure

$$C^\infty(Q/K) := C^\infty(Q)^K$$

(invariant smooth functions on Q). Thus, Q/K is a stratified topological space with smooth structure; see [16] for this notion.

The total space $\mathrm{T}^*(Q/K)$ is obtained as follows. The action of K on Q is lifted to a proper symplectic action of K on the cotangent bundle T^*Q by the corresponding point transformations. The map $J : \mathrm{T}^*Q \to \mathfrak{k}^*$ defined by

$$\langle J(\eta), X\rangle := \eta(X^Q), \quad \eta \in \mathrm{T}^*Q, X \in \mathfrak{k}, \tag{4}$$

where X^Q denotes the Killing vector field associated with X, is an equivariant momentum mapping for this action [1, Section 4.2]. (Thus, these data define a Hamiltonian G-manifold naturally associated with (Q, K).) Since J is equivariant, the level set $J^{-1}(0)$ is invariant under K. The bundle space $\mathrm{T}^*(Q/K)$ is given by the topological quotient $J^{-1}(0)/K$. It is equipped with the following structure; see [2,14,19] or [5, App. B.5]:

– A smooth Poisson structure. The natural smooth structure of $\mathrm{T}^*(Q/K)$ is given by

$$C^\infty\left(\mathrm{T}^*(Q/K)\right) := C^\infty(\mathrm{T}^*Q)^K/V^K,$$

where V denotes the vanishing ideal of the level set $J^{-1}(0)$ and V^K denotes the subset of K-invariants. Since K acts symplectically on T^*Q, $C^\infty(\mathrm{T}^*Q)^K$ is a Poisson subalgebra of $C^\infty(\mathrm{T}^*Q)$. In view of Noether's theorem, $J^{-1}(0)$ is invariant under the Hamiltonian flow of invariant functions. Hence, V^K is a Poisson ideal in $C^\infty(\mathrm{T}^*Q)^K$. Therefore, $C^\infty(\mathrm{T}^*(Q/K))$ inherits a Poisson bracket through

$$\{f + V^K, g + V^K\}_{\mathrm{T}^*(Q/K)} = \{f, g\}_{\mathrm{T}^*Q}, \qquad f, g \in C^\infty(\mathrm{T}^*(Q/K)).$$

– A stratification by orbit types. Using the slice theorem it can be shown that for given orbit type τ the subset $J^{-1}(0)_\tau$ of $J^{-1}(0)$ consisting of the elements of type τ is an embedded submanifold of T^*Q. Local charts on the τ-stratum $\mathrm{T}^*(Q/K)_\tau$ of $\mathrm{T}^*(Q/K)$ are then defined in the usual way: for a given point in $\mathrm{T}^*(Q/K)_\tau$ one chooses a representative in $J^{-1}(0)_\tau$ and a slice about the representative for the action of K on $J^{-1}(0)_\tau$. By restriction, the natural projection $\pi_\tau : J^{-1}(0)_\tau \to \mathrm{T}^*(Q/K)_\tau$ induces a homeomorphism of the slice onto its image. Thus, charts on the slice induce charts on $\mathrm{T}^*(Q/K)_\tau$.

– Symplectic structures on the strata $T^*(Q/K)_\tau$. One can prove that the annihilator of the pull-back of the symplectic form ω of T^*Q to the submanifold $J^{-1}(0)_\tau$ coincides with the distribution defined by the tangent spaces of the orbits. Therefore, the pull-back of ω to a slice for the action of K on $J^{-1}(0)_\tau$ is a symplectic form on that slice. Through the homeomorphism of the slice onto its image in $T^*(Q/K)_\tau$, induced by the natural projection π_τ, it defines a local symplectic form on $T^*(Q/K)_\tau$. Due to the fact that ω is K-invariant, all the local forms merge to a symplectic form ω^τ on $T^*(Q/K)_\tau$. Then

$$\pi_\tau^*\omega^\tau = j_\tau^*\omega,$$

where $j_\tau : J^{-1}(0)_\tau \to T^*Q$ denotes the natural injection.

By construction, the injections $(T^*Q)_\tau \to T^*(Q/K)$ are Poisson maps. Therefore, the above data turn $T^*(Q/K)$ into a stratified symplectic space.

Finally, the projection π of (3) is induced by the restriction of the natural (equivariant) projection $T^*Q \to Q$ to the level set $J^{-1}(0)$. Since $J^{-1}(0)$ contains the zero section of T^*Q, π is surjective.

Remark 3.1. The fibres of (3) may intersect several distinct strata of $T^*(Q/K)$. In particular, π does not preserve the orbit types. However, as the stabilizer of a covector in T^*Q cannot be larger than that of its base point, π does not decrease orbit types. For a detailed study of the stratifications of the fibres of $T^*(Q/K)$; see [15].

Remark 3.2. Since (3) is a bundle in the topological category in the sense of [9] and since it plays the same role for Q/K as the cotangent bundle T^*Q plays for Q, (3) is called the reduced cotangent bundle in [13], although in general its elements are not covectors. When K acts freely then Q/K is a manifold and (3) is isomorphic to the cotangent bundle of this manifold [1]. In general, the cotangent bundles of the strata of Q/K are dense subsets of the corresponding strata of $T^*(Q/K)$ [15].

If, like in our case, (Q, K) is the configuration space of a Hamiltonian system with symmetries, Q/K and $T^*(Q/K)$ are referred to as the reduced configuration space and the reduced phase space, respectively. It can be shown in general [14] that if an evolution curve in T^*Q w.r.t. a K-invariant Hamiltonian meets a submanifold $J^{-1}(0)_\tau$ then it is contained completely in this submanifold. Therefore, dynamics in $T^*(Q/K)$ takes place inside the strata. Due to Remark 3.1, an analogous statement for Q/K is in general not true, though.

We will now discuss the reduced data of our model in detail. The reduced configuration space Q/K and the reduced phase space $T^*(Q/K)$ will be denoted by $\mathcal{X}$ and $\mathcal{P}$, respectively.

3.2. The reduced configuration space $\mathcal{X}$

In what follows we will write G for SU(3) and $\mathfrak{g}$ for su(3).

By construction, $\mathcal{X}$ is the adjoint quotient G/Ad. As G is semisimple, this space has the following two standard realizations. Let T denote the subgroup of diagonal matrices of G. One has $T \cong \mathrm{U}(1) \times \mathrm{U}(1)$, a 2-torus. For $j = 1, 2, 3$, let $T_{(j)}$ denote the subsets of T consisting of the elements whose entries coincide, possibly except for the jth one. Let $\mathcal{A}$ denote one of the triangular subsets of T which are cut out by the $T_{(j)}$, $j = 1, 2, 3$; see Fig. 1. From the embedding $\mathcal{A} \to T$, $\mathcal{A}$ acquires a Whitney smooth structure. It is a standard fact that the embeddings $\mathcal{A} \to T \to G$ induce, by passing to quotients, isomorphisms

$$\mathcal{X} \cong T/S_3 \cong \mathcal{A} \tag{5}$$

of topological spaces with smooth structure. Here the symmetric group S_3 acts by permutation of entries and the smooth structure of T/S_3 is defined by the invariant smooth functions on T.

Let us describe the stratification. The number of distinct entries of $a \in \mathcal{A}$ can be 3, 2 or 1. Denote the corresponding subsets of $\mathcal{A}$ by $\mathcal{A}_k$ with $k = 2, 1, 0$. One has $\mathcal{A}_1 = \bigcup_{j=1}^3 \mathcal{A} \cap T_{(j)}$. Topologically, $\mathcal{A}$ is a 2-simplex, $\mathcal{A}_2$ is its interior, $\mathcal{A}_1$ consists of the edges without the vertices and $\mathcal{A}_0$ consists of the vertices. Taking into account that the stabilizer of a under the action of SU(3) is given by the centralizer of a in SU(3), the stabilizer of $a \in \mathcal{A}_k$ is

k	SU(3)-stabilizer	S_3-stabilizer
2	T	$\{\mathbb{1}\}$
1	U(2)	S_2
0	SU(3)	S_3

(6)

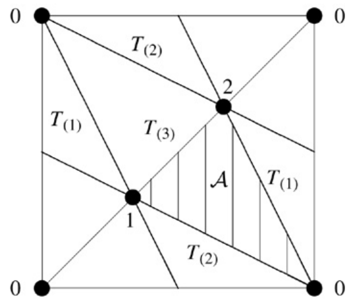

Fig. 1. A possible choice for the subset $\mathcal{A}$ of T. The numbers 0, 1, 2 stand for the central elements $\mathbb{1}$, $e^{i\frac{2}{3}\pi}\mathbb{1}$ and $e^{i\frac{4}{3}\pi}\mathbb{1}$, respectively.

In particular, $\mathcal{A}_0 = \mathbb{Z}_3$, the centre of SU(3). Denote the orbit types in the respective order by τ_2, τ_1 and τ_0, irrespective of the action they belong to, and the corresponding strata of $\mathcal{X}$ by $\mathcal{X}_2$, $\mathcal{X}_1$ and $\mathcal{X}_0$. (The numbering refers to the dimensions of the strata.) Type τ_2 is the principal orbit type and $\mathcal{X}_2$ is the principal stratum.

It is easy to see that the isomorphism $\mathcal{X} \cong \mathcal{A}$ holds on the level of stratified smooth topological spaces.

Remark 3.3. The identification of $\mathcal{X}$ with $\mathcal{A}$ endows $\mathcal{X}$ with a CW-complex structure in an obvious fashion. Already for the quotient $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mathrm{SU}(3)$ with SU(3) acting by diagonal conjugation, which is the reduced configuration space of lattice SU(3)-gauge theory on a lattice with 2 plaquettes, the construction of a CW-complex structure is much more complicated; see [4].

3.3. *The reduced phase space* $\mathcal{P}$

As anticipated in Section 2, we identify T^*G with the direct product $G \times \mathfrak{g}$ by virtue of the natural diffeomorphism

$$G \times \mathfrak{g} \to \mathrm{T}^*G, \qquad (a, A) \mapsto \langle A, \mathrm{R}'_{a^{-1}} \cdot \rangle. \tag{7}$$

Here, $\mathrm{R}_a : G \to G$ denotes right multiplication by $a \in G$ and $\langle \cdot, \cdot \rangle$ is the ordinary scalar product of complex matrices,

$$\langle A, B \rangle = \mathrm{tr}(A^\dagger B), \quad A, B \in \mathrm{M}_3(\mathbb{C}).$$

When restricted to $\mathfrak{g}$ this form yields a real scalar product which, up to a constant factor, coincides with the negative of the Killing form of $\mathfrak{g}$:

$$\langle A, B \rangle = -\mathrm{tr}(AB), \quad A, B \in \mathfrak{g}.$$

Since $\mathrm{T}(G \times \mathfrak{g}) \cong \mathrm{T}G \times \mathrm{T}\mathfrak{g}$, vectors tangent to $G \times \mathfrak{g}$ at (a, A) can be written as $(\mathrm{R}'_a B, (A, C))$ with $B, C \in \mathfrak{g}$. Under the identification (7) the symplectic potential of T^*G takes the standard form

$$\theta_{(a,A)}\left(R'_a B, (A, C)\right) = \langle A, B \rangle, \tag{8}$$

and hence the symplectic form $\omega = \mathrm{d}\theta$ is

$$\omega_{(a,A)}\left((R'_a B_1, (A, C_1)), (R'_a B_2, (A, C_2))\right) = \langle B_1, C_2 \rangle - \langle C_1, B_2 \rangle - \langle A, [B_1, B_2] \rangle. \tag{9}$$

The action of G on T^*G by the induced point transformations is given by conjugation, i.e.,

$$b \cdot (a, A) = (bab^{-1}, bAb^{-1}). \tag{10}$$

If we furthermore identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by virtue of the scalar product $\langle \cdot, \cdot \rangle$, the natural momentum mapping for this action is given by the map

$$J : G \times \mathfrak{g} \to \mathfrak{g}, \qquad J(a, A) = A - a^{-1}Aa. \tag{11}$$

The level set $J^{-1}(0)$ is therefore given by all pairs $(a, A) \in G \times \mathfrak{g}$ where a and A commute. In particular, it contains the subset $T \times \mathfrak{t}$. By restriction of the natural projection to orbits we obtain a map

$$\lambda : T \times \mathfrak{t} \to \mathcal{P}. \tag{12}$$

Let $(a, A) \in J^{-1}(0)$. Since a and A commute, they possess a common eigenbasis. Since a is unitary and A is anti-Hermitian, the eigenbasis can be chosen to be orthonormal. Hence, by G-action, (a, A) can be transported to $T \times \mathfrak{t}$. In other words, every G-orbit in $J^{-1}(0)$ intersects the subset $T \times \mathfrak{t}$. Hence, λ is surjective. Since two elements of $T \times \mathfrak{t}$ are conjugate under G iff they differ by a simultaneous permutation of their entries, then λ descends to a bijection

$$(T \times \mathfrak{t})/S_3 \to \mathcal{P}.$$

Standard arguments ensure that this is in fact a homeomorphism. Thus, we can use λ to describe $\mathcal{P}$. In particular, $\mathcal{P}$ is an orbifold.

We start with the stratification. The number of entries which simultaneously coincide for both a and A can be 0, 2 or 3. Denote the corresponding subsets of $T \times \mathfrak{t}$ by $(T \times \mathfrak{t})_k$ with $k = 2, 1, 0$, respectively. The stabilizers and orbit types of $(a, A) \in (T \times \mathfrak{t})_k$ under SU(3)-action and S_3-action are

k	SU(3)-stabilizer	S_3-stabilizer	orbit type
2	T	$\{\mathbb{1}\}$	τ_2
1	U(2)	S_2	τ_1
0	SU(3)	S_3	τ_0

(13)

Since the orbit types are the same as for $\mathcal{X}$ we use the same notation. Let $\mathcal{P}_k \subseteq \mathcal{P}$ denote the stratum of type τ_k, $k = 0, 1, 2$. $\mathcal{P}_2$ is the principal stratum. Since the subsets $(T \times \mathfrak{t})_k$ are the pre-images of the strata $\mathcal{P}_k$ under λ, they will be referred to as strata of $T \times \mathfrak{t}$. By restriction, λ induces maps

$$\lambda_k : (T \times \mathfrak{t})_k \to \mathcal{P}_k, \quad k = 2, 1, 0, \tag{14}$$

which descend to homeomorphisms of $(T \times \mathfrak{t})_k/S_3$ onto $\mathcal{P}_k$, $k = 2, 1, 0$.

We determine $(T \times \mathfrak{t})_k$ explicitly. Recall that $\mathbb{Z}_3$ denotes the centre of $G = \mathrm{SU}(3)$. As for $T_{(j)}$, let $\mathfrak{t}_{(j)}$, $j = 1, 2, 3$, denote the subset of $\mathfrak{t}$ consisting of the elements whose entries coincide, possibly except for the jth one. We find

$$\begin{aligned}
(T \times \mathfrak{t})_0 &= \mathbb{Z}_3 \times \{0\}, \\
(T \times \mathfrak{t})_1 &= \left(\bigcup_{j=1}^{3} T_{(j)} \times \mathfrak{t}_{(j)} \right) - (T \times \mathfrak{t})_0, \\
(T \times \mathfrak{t})_2 &= T \times \mathfrak{t} - (T \times \mathfrak{t})_1.
\end{aligned}$$

These are embedded submanifolds of $T \times \mathfrak{t}$. Since $\mathfrak{t}$ is the Lie subalgebra of $\mathfrak{g}$ associated with the Lie subgroup T of G, $T \times \mathfrak{t}$ is a *symplectic* submanifold of $G \times \mathfrak{g}$. Analogously, so are $T_{(j)} \times \mathfrak{t}_{(j)}$, $j = 1, 2, 3$. It follows that $(T \times \mathfrak{t})_k$, $k = 2, 1$, are symplectic manifolds. For convenience, in the following we will view $(T \times \mathfrak{t})_0$ as a (trivially) symplectic manifold, too.

Theorem 3.4. *The map λ is Poisson. The maps λ_k are local symplectomorphisms.*

Proof. By definition, $C^\infty(\mathcal{P})$ is a quotient of $C^\infty(G \times \mathfrak{g})^G$. Hence, the first assertion is a direct consequence of the fact that $T \times \mathfrak{t}$ is a symplectic submanifold of $G \times \mathfrak{g}$. For the second assertion, recall the construction of the symplectic forms on the strata $\mathcal{P}_k$ from Section 3.1. The assertion then follows by observing that any point of $\mathcal{P}_k$ has a representative in $(T \times \mathfrak{t})_k$ and that a sufficiently small neighbourhood of the chosen representative in $(T \times \mathfrak{t})_k$ provides a slice for the action of G on the submanifold $J^{-1}(0)_k$ of $G \times \mathfrak{g}$. Here $J^{-1}(0)_k$ denotes the subset of $J^{-1}(0)$ consisting of the elements of the orbits of type τ_k. □

Remark 3.5. 1. Since the submanifolds $(T \times \mathfrak{t})_k$ are symplectic and since S_3 is finite, the quotient $(T \times \mathfrak{t})/S_3$ naturally carries the structure of a stratified symplectic space. Of course, this structure might be viewed as to be obtained by singular Marsden–Weinstein reduction with (necessarily) trivial momentum map. Then Theorem 3.4 says that the map λ induces an isomorphism of stratified symplectic spaces of $(T \times \mathfrak{t})/S_3$ onto $\mathcal{P}$.

2. The dynamics on $\mathcal{P}$ is thus given by the dynamics on $T \times \mathfrak{t}$ w.r.t. an S_3-invariant Hamiltonian and the symplectic form (9). Similarly, motion on $\mathcal{X}$ is given by S_3-invariant motion on the 2-torus with metric defined by the scalar product $\langle \cdot, \cdot \rangle$.

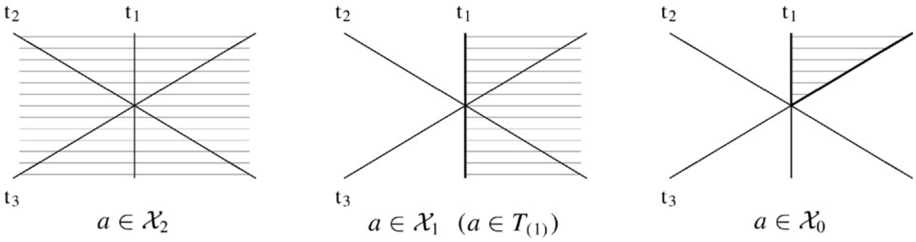

Fig. 2. The fibres $\pi^{-1}(a)$.

3.4. The projection $\pi : \mathcal{P} \to \mathcal{X}$

Recall from Section 3.1 that the projection $\pi : \mathcal{P} \to \mathcal{X}$ is induced by the cotangent bundle projection $\mathrm{T}^*G \to G$. By virtue of the identification (7), the latter is identified with the natural projection to the first factor $\mathrm{pr}_1 : G \times \mathfrak{g} \to G$. Hence, one has the following commutative diagram:

$$\begin{array}{ccc} T \times \mathfrak{t} & \xrightarrow{\lambda} & \mathcal{P} \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ T & \longrightarrow & \mathcal{X} \end{array}$$

where the lower horizontal arrow is defined by restriction of the natural projection $G \to \mathcal{X}$. It follows that the fibre over $a \in \mathcal{X}$ ($\mathcal{X}$ being identified with $\mathcal{A}$ and hence with a subset of T) is given by

$$\pi^{-1}(a) = \mathfrak{t}/S(a),$$

where $S(a)$ is the stabilizer of a under the action of S_3. According to (6), there are three cases, illustrated in Fig. 2.

- If $a \in \mathcal{X}_2$, $S(a)$ is trivial, and hence $\pi^{-1}(a) = \mathfrak{t}$. That is, the fibre is a full 2-plane and belongs to the stratum $\mathcal{P}_2$.
- If $a \in \mathcal{X}_1$ then $a \in T_{(j)} - \mathbb{Z}_3$ for some $j = 1, 2, 3$. Then $S(a) = S_2$, acting by permuting the two entries besides the jth one. Hence, $\pi^{-1}(a) = \mathfrak{t}/S_2$, acting by reflection about the subspace $\mathfrak{t}_{(j)}$. Therefore, the fibre may be identified with one of the two half-planes of $\mathfrak{t}$ cut out by $\mathfrak{t}_{(j)}$. Its interior belongs to the stratum $\mathcal{P}_2$, whereas the boundary $\mathfrak{t}_{(j)}$ belongs to the stratum $\mathcal{P}_1$.
- If $a \in \mathcal{X}_0$, i.e., $a \in \mathbb{Z}_3$, then $S(a) = S_3$. The action of S_3 on $\mathfrak{t}$ is generated by the reflections about the three subspaces $\mathfrak{t}_{(j)}$, $j = 1, 2, 3$. Hence, the fibre may be identified with one of the six (closed) Weyl chambers of $\mathfrak{t}$ cut out by $\mathfrak{t}_{(j)}$, $j = 1, 2, 3$ (the walls of the Weyl chambers). The interior of the Weyl chamber belongs to the stratum $\mathcal{P}_2$, the walls minus the origin belong to the stratum $\mathcal{P}_1$ and the origin belongs to the stratum $\mathcal{P}_0$.

One can see explicitly that the projection $\pi : \mathcal{P} \to \mathcal{X}$ does not preserve the stratification, because the fibres over points in $\mathcal{X}_1$ and $\mathcal{X}_0$ intersect more than one stratum of $\mathcal{P}$. As stated in Remark 3.1, this is a general phenomenon.

Remark 3.6. The shape of the fibres resembles the shape of a neighbourhood of the base point in $\mathcal{X}$. Indeed, the fibre over $a \in \mathcal{X}$ might be identified with the space of tangent vectors of smooth curves in $\mathcal{X}$ starting at a: for $a \in \mathcal{X}_2$, any tangent vector occurs; for $a \in \mathcal{X}_1$, the tangent vectors form a closed half-plane; and for $a \in \mathcal{X}_0$, they form a cone of angle $\pi/3$. Thus, intuitively the reduced phase may be identified with the tangent bundle of $\mathcal{X}$, defined in the above sense. This relation seems to be a general phenomenon in singular cotangent bundle reduction. It certainly deserves to be made precise, because it is likely to be the singular counterpart of the well-known result that, in the regular case, Marsden–Weinstein reduction of a cotangent bundle yields the cotangent bundle of the reduced base manifold.

Remark 3.7. The description of the reduced data given here generalizes to an arbitrary compact semisimple Lie group in an obvious way: T and $\mathfrak{t}$ are replaced by a maximal torus in G and its Lie algebra, which is a Cartan subalgebra of $\mathfrak{g}$. $\mathcal{A}$ is replaced by a Weyl alcove in T and S_3 is replaced by the Weyl group of G. It is interesting that for $G = \mathrm{SU}(2)$ one obtains the reduced phase space of the spherical pendulum with zero angular momentum, which is the well-known canoe [5, Section VI.2].

This completes the construction of the reduced data for the model under consideration. Next, we will derive tools for studying the dynamics of this model. That is, first, a symplectic covering of $T \times \mathfrak{t}$ and, second, a description of $\mathcal{P}$ and $\mathcal{X}$ in terms of invariants.

4. Symplectic covering of $T \times \mathfrak{t}$

Recall the symplectic form ω of $G \times \mathfrak{g}$; see (9). By an abuse of notation, the pull-back of this form to $T \times \mathfrak{t}$ will also be denoted by ω. Elements of $\mathbb{R}^4$ will be denoted by $(x, p) \equiv ((x^1, x^2), (p_1, p_2))$. In this section, we use the exponential map of T to construct a covering $\psi : \mathbb{R}^4 \to T \times \mathfrak{t}$ which pulls back ω to the natural symplectic form $\mathrm{d}p_i \wedge \mathrm{d}x^{\mathrm{i}}$ of $\mathbb{R}^4$ (summation convention). We choose ψ to be induced by some covering $\varphi : \mathbb{R}^2 \to T$ by virtue of the commutative diagram

$$\begin{array}{ccc} \mathrm{T}\mathbb{R}^2 & \xrightarrow{\varphi'} & \mathrm{T}T \\ g\big\downarrow & & \big\downarrow h \\ \mathbb{R}^4 \cong \mathrm{T}^*\mathbb{R}^2 & \xrightarrow{\psi} & \mathrm{T}^*T \cong T \times \mathfrak{t} \end{array} \tag{15}$$

where the vertical arrows stand for the isomorphisms between the tangent and cotangent bundles induced by the natural Riemannian metrics g on $\mathbb{R}^2$ and h on T. Recall that h is given by the restriction to T of the Killing metric of G induced by the scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. A straightforward calculation, where $\mathbb{R}^2$ and T may be replaced by arbitrary Riemannian manifolds, shows that if φ is isometric then ψ is symplectic. Thus, all we have to do is to choose φ appropriately. For example, we can choose φ as the composition of the isomorphism $\mathbb{R}^2 \to \mathfrak{t}$, mapping the canonical basis vectors e_1, e_2 to the orthonormal basis

$$\mathrm{diag}\left(\frac{\mathrm{i}}{\sqrt{6}}, \frac{\mathrm{i}}{\sqrt{6}}, -\mathrm{i}\sqrt{\frac{2}{3}}\right), \qquad \mathrm{diag}\left(\frac{\mathrm{i}}{\sqrt{2}}, -\frac{\mathrm{i}}{\sqrt{2}}, 0\right)$$

in $\mathfrak{t}$, with the exponential map $\mathfrak{t} \to T$:

$$\varphi(x) = \mathrm{diag}\left(\mathrm{e}^{\mathrm{i}\left(\frac{1}{\sqrt{6}}x^1 + \frac{1}{\sqrt{2}}x^2\right)}, \mathrm{e}^{\mathrm{i}\left(\frac{1}{\sqrt{6}}x^1 - \frac{1}{\sqrt{2}}x^2\right)}, \mathrm{e}^{-\mathrm{i}\sqrt{\frac{2}{3}}x^1}\right). \tag{16}$$

The corresponding covering $\psi : \mathbb{R}^4 \to T \times \mathfrak{t}$ is

$$\psi(x, p) = \left(\varphi(x), \mathrm{diag}\left(\mathrm{i}\left(\frac{1}{\sqrt{6}}p_1 + \frac{1}{\sqrt{2}}p_2\right), \mathrm{i}\left(\frac{1}{\sqrt{6}}p_1 - \frac{1}{\sqrt{2}}p_2\right), -\mathrm{i}\sqrt{\frac{2}{3}}p_1\right)\right). \tag{17}$$

Remark 4.1. Since ψ is a local diffeomorphism it is a local symplectomorphism and hence provides local Darboux coordinates on $T \times \mathfrak{t}$.

Now having constructed ψ, we can compose it with the map $\lambda : T \times \mathfrak{t} \to \mathcal{P}$, see (12), to obtain

$$\chi := \lambda \circ \psi : \mathbb{R}^4 \to \mathcal{P}. \tag{18}$$

Let $\mathbb{R}^4_k = \chi^{-1}(\mathcal{P}_k)$ denote the pre-image of the stratum $\mathcal{P}_k$ under χ, $k = 2, 1, 0$. Using $\mathbb{R}^4_k = \psi^{-1}((T \times \mathfrak{t})_k)$ we find

$$\mathbb{R}^4_0 = \mathbb{R}^2_0 \times \{0\}, \qquad \mathbb{R}^4_1 = \left(\bigcup_{j=1}^{3} \bigcup_{l \in \mathbb{Z}} \mathbb{R}^2_{(j)l} \times \mathbb{R}^2_{(j)0}\right) \setminus \mathbb{R}^4_0, \qquad \mathbb{R}^4_2 = \mathbb{R}^4 \setminus \mathbb{R}^4_1, \tag{19}$$

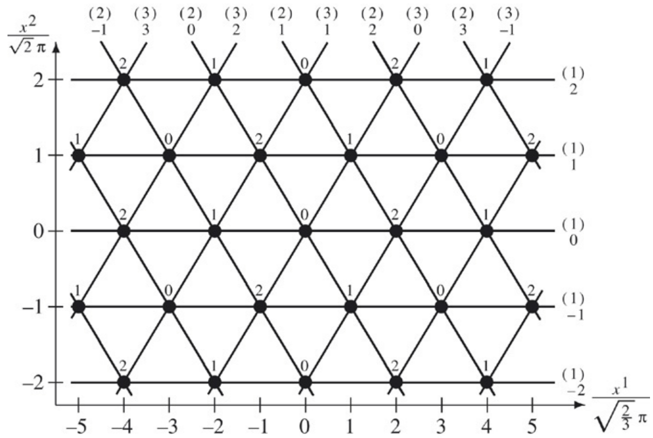

Fig. 3. The subsets $\mathbb{R}^2_0$ and $\mathbb{R}^2_{(j)l}$ of $\mathbb{R}^2$. The elements of $\mathbb{R}^2_0$ are represented by • and are labelled by the element of $\mathcal{X}_0$ they project to: 0, 1, 2 stands for $\mathbb{1}$, $\exp \mathrm{i}\frac{2}{3}\pi\mathbb{1}$, $\exp \mathrm{i}\frac{4}{3}\pi\mathbb{1}$, respectively. The affine subspaces $\mathbb{R}^2_{(j)l}$ are labelled by $\binom{(j)}{l}$.

where

$$\mathbb{R}^2_0 = \left\{ \left(l\sqrt{\frac{2}{3}}\pi, (l+2m)\sqrt{2}\pi \right) \middle| \, l, m \in \mathbb{Z} \right\}$$
$$\mathbb{R}^2_{(1)l} = \left\{ \left(y, \sqrt{3}y + 2l\sqrt{2}\pi \right) \middle| \, y \in \mathbb{R} \right\}$$
$$\mathbb{R}^2_{(2)l} = \left\{ \left(y, -\sqrt{3}y + 2l\sqrt{2}\pi \right) \middle| \, y \in \mathbb{R} \right\}$$
$$\mathbb{R}^2_{(3)l} = \left\{ \left(y, l\sqrt{2}\pi \right) \middle| \, y \in \mathbb{R} \right\}.$$

The $\mathbb{R}^2_{(j)l}$ are affine subspaces of $\mathbb{R}^2$, intersecting each other in the points of $\mathbb{R}^2_0$; see Fig. 3. The $\mathbb{R}^4_k$ are symplectic submanifolds of $\mathbb{R}^4$: for $k = 0$ this is trivial, for $k = 2$ it is obvious. For $k = 1$ it follows from the fact that in the natural identification of $\mathrm{T}^*\mathbb{R}^2$ with $\mathbb{R}^4$ utilized here, $\mathbb{R}^2_{(j)l} \times \mathbb{R}^2_{(j)l}$ corresponds to $\mathrm{T}^*\mathbb{R}^2_{(j)l}$, $j = 1, 2, 3$, $l \in \mathbb{Z}$.

By restriction, ψ and χ induce maps

$$\psi_k : \mathbb{R}^4_k \to (T \times \mathfrak{t})_k, \qquad \chi_k = \lambda_k \circ \psi_k : \mathbb{R}^4_k \to \mathcal{P}_k, \quad k = 2, 1, 0, \tag{20}$$

respectively.

Theorem 4.2. *The map χ is Poisson, i.e., for $f, g \in C^\infty(\mathcal{P})$ there holds*

$$\chi^*\{f, g\}_{\mathcal{P}} = \frac{\partial(\chi^* f)}{\partial x^k}\frac{\partial(\chi^* g)}{\partial p_k} - \frac{\partial(\chi^* f)}{\partial p_k}\frac{\partial(\chi^* g)}{\partial x^k}.$$

The maps χ_k are local symplectomorphisms.

Proof. This follows from Theorem 3.4. In addition, for the second assertion one has to use that the ψ_k are local symplectomorphisms. This is a consequence of the fact that $(T \times \mathfrak{t})_k$ are embedded submanifolds of $T \times \mathfrak{t}$. □

5. Description in terms of invariants

In this section, we derive the invariant-theoretic description of the reduced data of our model. Let us start with recalling the general theory. Consider an orthogonal representation of some Lie group H on a Euclidean space $\mathbb{R}^n$. The algebra of invariant polynomials of this representation is finitely generated [20]. Any finite set of generators $\rho_1, \ldots, \rho_p$ defines a map

$$\rho = (\rho_1, \ldots, \rho_p) : \mathbb{R}^n/H \to \mathbb{R}^p.$$

This map is a homeomorphism onto its image [18] and the image is a closed semialgebraic subset of $\mathbb{R}^p$, i.e., it is the solution set of a logical combination of algebraic equations and inequalities. The equations are provided by the relations amongst the generators ρ_i and the inequalities keep track of their ranges. The set $\{\rho_1, \dots, \rho_p\}$ and the map ρ are called a Hilbert basis and a Hilbert map for the representation, respectively. If $V \subseteq \mathbb{R}^n$ is an H-invariant semialgebraic subset, then ρ restricts to a homeomorphism of V/H onto the image $\rho(V) \subseteq \mathbb{R}^p$ and the image is again a semialgebraic subset. The equations are now given by the relations amongst the restricted mappings $\rho_i|_V$ and the inequalities are given by their ranges.

5.1. Hilbert map

To apply the method explained above to our model, we consider the realification of the representation of $G = \mathrm{SU}(3)$ on $\mathrm{M}_3(\mathbb{C}) \oplus \mathrm{M}_3(\mathbb{C})$ by diagonal conjugation:

$$a \cdot (X_1, X_2) = (aX_1a^{-1}, aX_2a^{-1}) \tag{21}$$

and set $V = J^{-1}(0) \subseteq G \times \mathfrak{g}$. Indeed, since this (complex) representation is unitary w.r.t. the scalar product

$$\langle (X_1, X_2), (Y_1, Y_2)\rangle = \mathrm{tr}(X_1^\dagger Y_1) + \mathrm{tr}(X_2^\dagger Y_2), \tag{22}$$

the realification, equipped with the real part of (22) as a scalar product, is orthogonal. Moreover, the subset $J^{-1}(0) \subseteq \mathrm{M}_3(\mathbb{C}) \oplus \mathrm{M}_3(\mathbb{C})$ is defined by the equations

$$a^\dagger a = \mathbb{1}, \qquad \det(a) = 1, \qquad A^\dagger + A = 0, \qquad aA - Aa = 0, \tag{23}$$

and hence is real algebraic.

Since the invariant polynomials of the realification of a complex representation are given by the real and imaginary parts of the invariant polynomials of the original representation, we have to find the generators for the latter. According to [17], a set of generators for the invariant polynomials of the representation of $\mathrm{SU}(n)$ on $\mathrm{M}_n(\mathbb{C})^m$ by diagonal conjugation is given by the trace monomials up to order $2^n - 1$ in $X_1, \dots, X_m$ and $X_1^\dagger, \dots, X_m^\dagger$. The generators are subject to the relation

$$\sum_{\sigma \in S_{n+1}} \mathrm{sgn}(\sigma) \prod_{\substack{(k_1,\dots,k_j) \\ \text{cycle of } \sigma}} \mathrm{tr}(Y_{k_1} \cdots Y_{k_j}) = 0, \qquad Y_1, \dots, Y_{n+1} \in \mathrm{M}_n(\mathbb{C}), \tag{24}$$

called the fundamental trace identity (FTI). Thus, according to the general theory, the real and imaginary parts of the trace monomials up to order 7 in a, A and $a^\dagger$, $A^\dagger$, where $(a, A) \in J^{-1}(0)$, provide a homeomorphism of $\mathcal{P}$ onto a semialgebraic subset of $\mathbb{R}^p$ for some large p. However, for the restrictions of the trace monomials to $J^{-1}(0)$ more relations hold than just the FTI. We can use them to reduce the set of generators and thus to simplify the Hilbert map. They arise from the matrix identities (23) and the Cayley–Hamilton theorem which says that the characteristic polynomial χ_X of any $X \in \mathrm{M}_n(\mathbb{C})$ obeys $\chi_X(X) = 0$. The characteristic polynomials of a and A are

$$\chi_a(z) = -z^3 + \mathrm{tr}(a)z^2 - \overline{\mathrm{tr}(a)}z + 1, \qquad \chi_A(z) = -z^3 + \frac{1}{2}\mathrm{tr}(A^2)z + \frac{1}{3}\mathrm{tr}(A^3), \tag{25}$$

respectively. Using (23), any trace monomial can be transformed to the form $\mathrm{tr}(a^k A^l)$ or its conjugate for some k, l. Using (25) it can then be rewritten as a polynomial in the monomials

$$\mathrm{tr}(a), \qquad \mathrm{tr}(aA), \qquad \mathrm{tr}(aA^2), \qquad \mathrm{tr}(A^2), \qquad \mathrm{tr}(A^3).$$

We define

$$c_k := \mathrm{Re}(\mathrm{tr}(a(-\mathrm{i}A)^k)), \qquad d_k := \mathrm{Im}(\mathrm{tr}(a(-\mathrm{i}A)^k)), \quad k = 0, 1, 2,$$
$$t_k := \mathrm{tr}((-\mathrm{i}A)^k), \quad k = 2, 3.$$

As $\mathrm{i}A$ is self-adjoint, t_2 and t_3 are real. Thus, we arrive at the simplified Hilbert map

$$\rho_{\mathcal{P}} = (c_0, d_0, c_1, d_1, c_2, d_2, t_2, t_3) : \mathcal{P} \to \mathbb{R}^8.$$

By embedding $G \hookrightarrow G \times \{0\} \subseteq J^{-1}(0)$, from $\rho_{\mathcal{P}}$ we obtain the Hilbert map for the action of G on itself by inner automorphisms, i.e., for the reduced configuration space $\mathcal{X}$:

$$\rho_{\mathcal{X}} = (c_0, d_0) : \mathcal{X} \to \mathbb{R}^2. \tag{26}$$

Analogously, embedding $\mathfrak{g} \hookrightarrow \{\mathbb{1}\} \times \mathfrak{g} \subseteq J^{-1}(0)$ and using that on the image of this embedding there holds $c_2 = t_2$ and $c_1 = d_1 = d_2 = 0$, we obtain the Hilbert map for the adjoint representation of SU(3), or the corresponding representation of S_3 on $\mathfrak{t}$,

$$\rho_{\mathrm{Ad}} = (t_2, t_3) : \mathrm{su}(3)/\mathrm{Ad} \to \mathbb{R}^2.$$

By construction, the maps $\rho_{\mathcal{P}}$, $\rho_{\mathcal{X}}$ and ρ_{Ad} are homeomorphisms onto their images. The images will be denoted by $\tilde{\mathcal{P}}$, $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$, respectively. The images of the strata $\mathcal{P}_k$ of $\mathcal{P}$ and $\mathcal{X}_k$ of $\mathcal{X}$ will be denoted by $\tilde{\mathcal{P}}_k$ and $\tilde{\mathcal{X}}_k$, respectively. As $\tilde{\mathcal{P}}$, $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$ are projections of a semialgebraic subset, they are semialgebraic themselves. The reason why we consider $\tilde{\mathcal{Y}}$ is that it will be needed in the discussion of $\tilde{\mathcal{P}}$.

5.2. Reduced configuration space and quotient of adjoint representation

The subset $\tilde{\mathcal{X}}$ was discussed in [3]. We recall the results. A natural candidate for an inequality is given by the discriminant $\mathrm{D}(\chi_a)$ of χ_a. Indeed, as a has eigenvalues α, β and $\overline{\alpha\beta}$, where $\alpha, \beta \in \mathrm{U}(1)$,

$$\mathrm{D}(\chi_a) = (\alpha - \beta)^2(\alpha - \overline{\alpha\beta})^2(\beta - \overline{\alpha\beta})^2 = -|\alpha\beta|^2|\alpha - \beta|^2|\alpha - \overline{\alpha\beta}|^2|\beta - \overline{\alpha\beta}|^2 \leq 0.$$

Define

$$P_1(c_0(a), d_0(a)) := -\mathrm{D}(\chi_a), \quad a \in \mathrm{SU}(3).$$

Expressing the discriminant in terms of the coefficients of χ_a, see (25), yields

$$P_1(c_0, d_0) = 27 - c_0^4 - 2c_0^2 d_0^2 - d_0^4 + 8c_0^3 - 24c_0 d_0^2 - 18c_0^2 - 18d_0^2.$$

Moreover, define

$$P_0(c_0, d_0) := 9 - c_0^2 - d_0^2.$$

Theorem 5.1. *$\tilde{\mathcal{X}}$ is the subset of $\mathbb{R}^2$ defined by the inequality $P_1 \geq 0$. As subsets of $\tilde{\mathcal{X}}$, the strata are defined by the following equations and inequalities:*

$$\tilde{\mathcal{X}}_0\colon P_0 = 0, \qquad \tilde{\mathcal{X}}_1\colon P_1 = 0 \text{ and } P_0 > 0, \qquad \tilde{\mathcal{X}}_2\colon P_1 > 0.$$

Proof. By construction, $\tilde{\mathcal{X}}$ is contained in the subset defined by $P_1 \geq 0$. The inverse inclusion was shown in [3]. To discuss the stratification, let $a \in \mathcal{X}$ (again identified with $\mathcal{A}$). One has $a \in \mathcal{X}_2$ iff all its entries are distinct, i.e., iff $\mathrm{D}(\chi_a) \neq 0$. This yields the assertion for $\tilde{\mathcal{X}}_2$. On has $a \in \mathcal{X}_0$ iff all its entries are equal. This is equivalent to $|\mathrm{tr}(a)| = 3$, i.e., $c_0(a)^2 + d_0^2(a) = 9$, and hence the assertion for $\tilde{\mathcal{X}}_0$. Then the assertion for $\tilde{\mathcal{X}}_1$ follows. □

The curve $P_1 = 0$ is a 3-hypocycloid in a circle of radius 3 and $\tilde{\mathcal{X}}$ is the subset of $\mathbb{R}^2$ enclosed by this hypocycloid; see Fig. 4.

Next, consider $\tilde{\mathcal{Y}}$. Again, the discriminant of χ_A is a natural candidate for an inequality: as A has purely imaginary eigenvalues, $\mathrm{D}(\chi_A) \leq 0$. Define

$$P_2(t_2(A), t_3(A)) := -\mathrm{D}(\chi_A), \quad A \in \mathrm{su}(3).$$

In terms of the coefficients of χ_A,

$$P_2(t_2, t_3) = \frac{1}{2}t_2^3 - 3t_3^2.$$

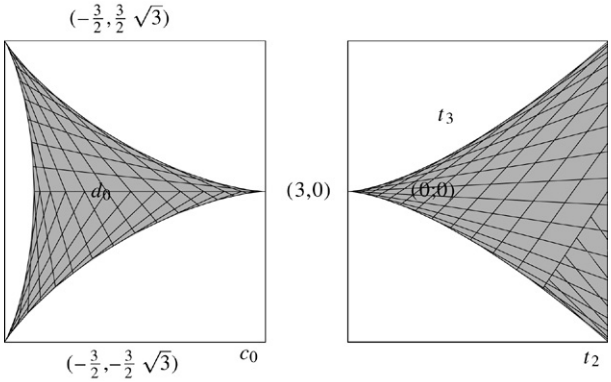

Fig. 4. The subsets $P_1 \geq 0$ (left) and $P_2 \geq 0$ (right). The curve $P_1 = 0$ is a 3-hypocycloid. All the singular points of the curves $P_1 = 0$ and $P_2 = 0$ are cusps.

Lemma 5.2. *$\tilde{\mathcal{Y}}$ is the subset of $\mathbb{R}^2$ defined by the inequality $P_2 \geq 0$. A matrix $A \in \mathfrak{t}$ has n distinct entries iff the following conditions hold:*

$$n = 1\colon t_2 = 0, \qquad n = 2\colon P_2 = 0 \text{ and } t_2 > 0, \qquad n = 3\colon P_2 > 0.$$

Proof. By construction, $\tilde{\mathcal{Y}}$ is contained in the subset of $\mathbb{R}^2$ defined by $P_2 \geq 0$. Conversely, for any choice of $(t_2, t_3) \in \mathbb{R}^2$ there exists $A \in \mathrm{M}_3(\mathbb{C})$ with these values for the invariants t_2, t_3. It may be chosen as a diagonal matrix with entries being the zeros of the polynomial χ_A, see (25), where the traces have to be expressed in terms of the chosen values for t_2 and t_3. It suffices to show that the inequality $P_2(t_2, t_3) \geq 0$ implies $A \in \mathfrak{g} = \mathrm{su}(3)$. Indeed, replacing z by $\mathrm{i}w$ yields $\mathrm{i}\chi_A = -w^3 + \frac{1}{2}t_2 w + \frac{1}{3}t_3$. This polynomial has real coefficients and discriminant $-P_2(t_2, t_3) \leq 0$. Therefore, its roots w_1, w_2, w_3 are real. Since it does not contain a square term, $w_1 + w_2 + w_3 = 0$. Then $A = \mathrm{diag}(\mathrm{i}w_1, \mathrm{i}w_2, \mathrm{i}w_3) \in \mathrm{su}(3)$.

The conditions that all entries are equal or that all entries are distinct are obvious. The condition that two entries are distinct then follows on observing that $P_2 \geq 0$ implies $t_2 \geq 0$. □

The curve $P_2 = 0$ is shown in Fig. 4. The inequality $P_2 \geq 0$ describes the part of the t_2–t_3 plane to the right of this curve.

5.3. Reduced phase space

Now we turn to $\tilde{\mathcal{P}}$. First, let us look for equations defining $J^{-1}(0)$ inside $G \times \mathfrak{g}$, i.e., reflecting the fact that a and A commute. The following two families of functions on $G \times \mathfrak{g}$ obviously vanish on $J^{-1}(0)$:

$$f_k(a, A) := 2(-\mathrm{i})^{k-1}(\mathrm{tr}(A^k a A a^\dagger) - \mathrm{tr}(A^{k+1})),$$
$$g_k(a, A) := (-\mathrm{i})^{k-1}(\mathrm{tr}(A^k a A a) - \mathrm{tr}(A^{k+1} a^2)), \quad k = 1, 2, \ldots.$$

The f_k and g_k are polynomials on $G \times \mathfrak{g}$. The f_k have real coefficients and the g_k have complex coefficients. Being invariant, they can be written as polynomials in the variables c_k, d_k, t_k. This way, we obtain two families of equations whose common zero set contains $\rho(\mathcal{P})$. They cannot all be independent. Indeed, for $k \geq 3$, using (25) one finds

$$f_k(a, A) = -\frac{1}{2}\mathrm{tr}(A^2) f_{k-2}(a, A) + \frac{\mathrm{i}}{3}\mathrm{tr}(A^3) f_{k-3}(a, A),$$
$$g_k(a, A) = -\frac{1}{2}\mathrm{tr}(A^2) g_{k-2}(a, A) + \frac{\mathrm{i}}{3}\mathrm{tr}(A^3) g_{k-3}(a, A),$$

where $f_0 = g_0 \equiv 0$. Hence, the relevant equations are those arising from f_1, f_2, g_1 and g_2. Taking the real and imaginary parts – f_1 and f_2 are already real — we obtain the following six equations:

$$f_1 = (3 + c_0^2 + d_0^2)t_2 - 2(c_1^2 + d_1^2) - 4(c_0c_2 + d_0d_2) = 0, \tag{27}$$

$$f_2 = \left(3 - \frac{1}{3}(c_0^2 + d_0^2)\right) t_3 - 2(c_1c_2 + d_1d_2) = 0, \tag{28}$$

$$\mathrm{Re}(g_1) = c_0c_2 - d_0d_2 - 2c_0t_2 - c_1^2 + d_1^2 + 3c_2 = 0, \tag{29}$$

$$\mathrm{Im}(g_1) = c_0d_2 + d_0c_2 + 2d_0t_2 - 2c_1d_1 - 3d_2 = 0, \tag{30}$$

$$\mathrm{Re}(g_2) = \frac{1}{2}((c_0 + 1)c_1 - d_0d_1)t_2 + \left(\frac{1}{3}(c_0^2 - d_0^2) - c_0\right) t_3 - c_1c_2 + d_1d_2 = 0, \tag{31}$$

$$\mathrm{Im}(g_2) = \frac{1}{2}((c_0 - 1)d_1 + d_0c_1)t_2 + \left(\frac{2}{3}c_0d_0 + d_0\right) t_3 - c_1d_2 - d_1c_2 = 0. \tag{32}$$

These are the candidates for the equations defining $\tilde{\mathcal{P}}$.

Next, we look for the inequalities. Besides the two inequalities $P_1 \geq 0$ and $P_2 \geq 0$ found above, which contain only pure invariants, there is another obvious one which contains the mixed invariants c_2 and d_2. Namely, for given $a \in T$ and $A \in \mathfrak{t}$, the entries of $a(-\mathrm{i}A)^2$ are complex numbers whose modulus is given by the corresponding entry of $(-\mathrm{i}A)^2$. Hence, $|\mathrm{tr}(a(-\mathrm{i}A)^2)| \leq \mathrm{tr}((-\mathrm{i}A)^2)$. In terms of the real invariants this reads $P_3(c_2, d_2, t_2) \geq 0$, where

$$P_3(c_2, d_2, t_2) := t_2^2 - c_2^2 - d_2^2.$$

Theorem 5.3. *$\tilde{\mathcal{P}}$ is the subset of $\mathbb{R}^8$ defined by the equations and inequalities*

$$f_1 = f_2 = \mathrm{Re}(g_1) = \mathrm{Im}(g_1) = \mathrm{Im}(g_2) = 0, \qquad P_j \geq 0, \quad j = 1, 2, 3. \tag{33}$$

Proof. We have already checked that $\tilde{\mathcal{P}}$ is contained in the subset (33). In order to prove the inverse inclusion, let there be given a point $x = (c_0, d_0, c_1, d_1, c_2, d_2, t_2, t_3)$ from the subset (33). We have to show that there exists a pair $(a, A) \in T \times \mathfrak{t}$ such that $\rho_{\mathcal{P}}(a, A) = x$. Due to Theorem 5.1 and Lemma 5.2, there exist $a \in T$ and $A \in \mathfrak{t}$ with $\rho_{\mathcal{X}}(a) = (c_0, d_0)$ and $\rho_{\mathrm{Ad}}(A) = (t_2, t_3)$, respectively. All pairs in the orbit of (a, A) under the direct product action of $S_3 \times S_3$ on $T \times \mathfrak{t}$ have the same values for the invariants c_0, d_0, t_2, t_3. Hence, if in (33) we view c_0, d_0, t_2, t_3 as fixed parameters and c_1, d_1, c_2, d_2 as the variables, it suffices to show that the number n_{sol} of distinct solutions of this system of equations and inequalities does not exceed the number n_{orb} of orbits under the diagonal action of S_3 on the $S_3 \times S_3$-orbit of (a, A). This holds in particular if $n_{\mathrm{sol}} = 1$, i.e., if the solution is unique.

We start with separating c_2 and d_2 in the equations $\mathrm{Re}(g_1) = 0$ and $\mathrm{Im}(g_1) = 0$:

$$P_0c_2 = (3 - c_0)c_1^2 - (3 - c_0)d_1^2 - 2d_0c_1d_1 + 2(c_0(3 - c_0) + d_0^2)t_2, \tag{34}$$

$$P_0d_2 = d_0c_1^2 - d_0d_1^2 - 2(3 + c_0)c_1d_1 + 2d_0(3 + 2c_0)t_2. \tag{35}$$

The inequality $P_1 \geq 0$ allows for three values of c_0, d_0 where the factor P_0 vanishes:

$$(c_0, d_0) = (3, 0), \left(-\frac{3}{2}, \frac{3}{2}\sqrt{3}\right), \left(-\frac{3}{2}, -\frac{3}{2}\sqrt{3}\right).$$

In the first case, the combination $f_1 + 2\,\mathrm{Re}(g_1) = 0$ yields $c_1 = 0$. Then (29) reads $6(t_2 - c_2) + d_1^2 = 0$ and (30) reads $d_1(t_2 - c_2) = 0$. It follows that $d_1 = 0$ and $c_2 = t_2$. Then $P_3 \geq 0$ implies $d_2 = 0$. In the other two cases, $f_1 - 4\,\mathrm{Re}(g_1) = 0$ implies $c_1 = d_1 = 0$. Resolving f_1 for c_2 and inserting this into P_3 yields $-(\sqrt{3}t_2 \pm 2d_2)^2 \geq 0$. Hence, $d_2 = \mp\frac{\sqrt{3}}{2}t_2$ and, then, $c_2 = -\frac{1}{2}t_2$. In all three cases $n_{\mathrm{sol}} = 1$.

For the rest of the proof assume $P_0 \neq 0$ (due to $P_1 \geq 0$ then $P_0 > 0$). Then c_2 and d_2 are fixed by (34) and (35) and can be replaced in (27) and (28):

$$2(9 + 6c_0 - 3c_0^2 + d_0^2)c_1^2 + 2Q_1d_1^2 - 8d_0(3 + 2c_0)c_1d_1 - P_1t_2 = 0, \tag{36}$$

$$\begin{aligned} &2(c_0 - 3)c_1^3 + 2d_0d_1^3 + 2d_0c_1^2d_1 + 2(9 + c_0)c_1d_1^2 + 4(c_0^2 - 3c_0 - d_0^2)t_2c_1 \\ &\quad - (12 + 8c_0)d_0t_2d_1 + \frac{1}{3}P_0^2t_3 = 0, \end{aligned} \tag{37}$$

where we have introduced the notation

$$Q_1 = (3 - c_0)^2 - 3d_0^2.$$

The coefficient Q_1 vanishes exactly for the three values of c_0, d_0 which obey $P_0 = 0$. Hence, we can solve (36) for d_1,

$$d_1 = \frac{1}{2Q_1}\left((12 + 8c_0)d_0c_1 \pm \sqrt{2P_1(Q_1t_2 - 6c_1^2)}\right). \tag{38}$$

If $t_2 = 0$ then $c_1 = 0$, because d_1 must be real, and hence $d_1 = 0$. Due to $P_2 \geq 0$, also $t_3 = 0$. Then (34) and (35) imply $c_2 = d_2 = 0$. Thus, again $n_{\text{sol}} = 1$.

In the sequel assume $t_2 \neq 0$ (due to $P_2 \geq 0$ then $t_2 > 0$). If $P_1 = 0$, d_1 is a multiple of c_1, and hence replacing d_1 in (37) yields a third-order polynomial equation which has at most three real solutions. That is, $n_{\text{sol}} \leq 3$. On the other hand, due to Theorem 5.1, a has two distinct entries. Due to Lemma 5.2, A has at least two distinct entries. Therefore, $n_{\text{orb}} = 3$.

In what follows we assume $P_1(c_0, d_0) > 0$. Then a has three distinct eigenvalues.

First, consider the case $d_0 = 0$. Here, d_1 is a pure root and (37) contains d_1 only in second order. Hence, inserting (38) and discarding the global factor $\frac{(c_0+3)^2}{3(c_0-3)}$ we obtain the third-order polynomial equation

$$24c_1^3 - 3(3 - c_0)^2t_2c_1 - (3 - c_0)^3t_3 = 0. \tag{39}$$

Since this equation has at most three real roots, each of which gives rise to at most two values of d_1 by (38), $n_{\text{sol}} \leq 6$. It follows that in the case $P_2 > 0$, where A has three distinct eigenvalues, $n_{\text{orb}} = 6 \geq n_{\text{sol}}$. In the case $P_2 = 0$, A has two distinct eigenvalues, so that $n_{\text{orb}} = 3$. To determine n_{sol} for this case, set

$$x := \sqrt[3]{\frac{t_3}{6}}.$$

Then $t_2 = 6x^2$ and $t_3 = 6x^3$. Substituting this in (39) and dividing by 6 we obtain

$$4c_1^3 - 3(3 - c_0)^2x^2c_1 - (3 - c_0)^3x^3 = 0.$$

Since $x \neq 0$ by assumption, the solutions of this equation are given by $c_1 = \tilde{c}_1x$, where $\tilde{c}_1$ are the solutions of the same equation with $x = 1$. We find $\tilde{c}_1 = 3 - c_0$ with multiplicity 1 and $\tilde{c}_1 = \frac{1}{2}(c_0 - 3)$ with multiplicity 2. Then (38) yields $d_1 = 0$ in the first case and $d_1 = \pm\frac{3}{2}\sqrt{(3 - c_0)(1 + c_0)}x$ in the second one. Thus, $n_{\text{sol}} = 3 = n_{\text{orb}}$.

Next, consider the case $d_0 \neq 0$. We insert (38) into (37) and write this equation in the form

$$\pm 3d_0(Q_1t_2 - 24c_1^2)\sqrt{2P_1(Q_1t_2 - 6c_1^2)} = Q, \tag{40}$$

where Q is some polynomial and we have omitted a common factor $3\sqrt{2}P_0^2/Q_1^3$ to avoid fractures. By squaring (40) we obtain the sixth-order polynomial equation in c_1

$$1152c_1^6 - 288Q_1t_2c_1^4 + 96Q_2t_3c_1^3 + 18Q_1^2t_2^2c_1^2 - 12Q_3t_2t_3c_1 + 2Q_1^3t_3^2 - 9P_1d_0^2t_2^3 = 0, \tag{41}$$

where

$$Q_2 = c_0^3 + 9c_0d_0^2 - 9c_0^2 + 27d_0^2 + 27c_0 - 27,$$
$$Q_3 = c_0^5 + 6c_0^3d_0^2 - 27c_0d_0^4 - 15c_0^4 - 81d_0^4 + 90c_0^3 - 162c_0d_0^2 - 270c_0^2 + 324d_0^2 + 405c_0 - 243,$$

and we have omitted a global factor Q_1^3. To a solution c_1 of (41) for which the l.h.s. of (40) does not vanish there corresponds one of the two signs in (40) and hence by (38) a unique value for d_1. To a solution for which $Q_1t_2 - 6c_1^2 = 0$ there corresponds a unique d_1 anyway. To a solution for which $Q_1t_2 - 24c_1^2 = 0$ there correspond two values of d_1, but such a solution necessarily has multiplicity 2. (This phenomenon should be interpreted the other way around: generically (41) has distinct solutions c_1, each with its own associated d_1. When two of the solutions happen to coincide, the associated values of d_1 seem to emerge from the same c_1.) From these observations we conclude that $n_{\text{sol}} \leq 6$, so that for $P_2 > 0$ we have $n_{\text{orb}} = 6 \geq n_{\text{sol}}$.

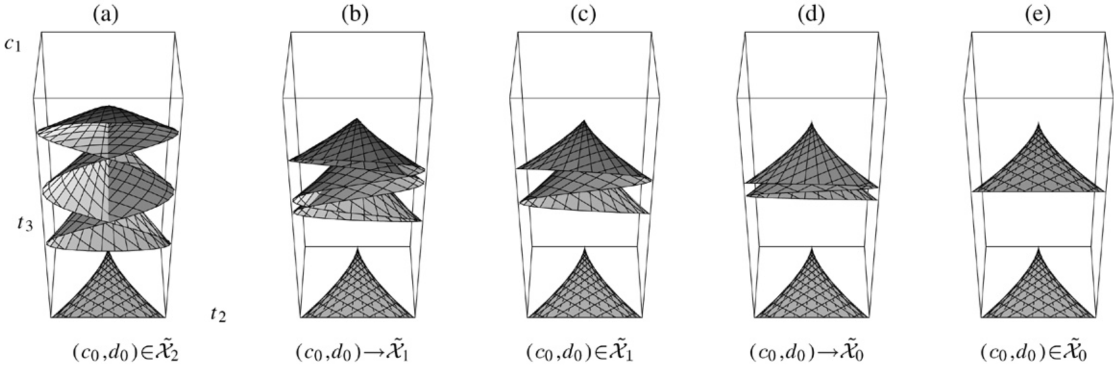

Fig. 5. Projection of the fibres $\tilde{\pi}^{-1}(c_0, d_0)$ to the t_2–t_3–c_1 plane (top) and the t_2–t_3 plane (bottom).

It remains to consider the case $P_2 = 0$, where $n_{\mathrm{orb}} = 3$. As before, we replace $t_2 = 6x^2$ and $t_3 = 6x^3$ in (41) and argue that the solutions of the resulting equation are given by $c_1 = \tilde{c}_1 x$, where $\tilde{c}_1$ are the solutions of this equation with x set to 1. The latter equation turns out to be the square of

$$4\tilde{c}_1^3 - 3Q_1\tilde{c}_1 + Q_2 = 0, \tag{42}$$

and hence it has at most three distinct real solutions $\tilde{c}_1$. We claim that for none of the corresponding solutions $c_1 = \tilde{c}_1 x$ does the factor $Q_1 t_2 - 24c_1^2 = 6x(Q_1 - 4\tilde{c}_1^2)$ in (40) vanish. Assume, on the contrary, $Q_1 - 4\tilde{c}_1^2 = 0$. Inserting $\tilde{c}_1 = \pm\sqrt{Q_1}$ into (42) and separating the terms with the root yields $\pm Q_1\sqrt{Q_1} = Q_2$. Taking the square we obtain $27d_0^2 P_1 = 0$, in contradiction to the assumptions $d_0 \neq 0$ and $P_1 \neq 0$. It follows that to each c_1 there corresponds a unique value for d_1. Thus, $n_{\mathrm{sol}} = 3 = n_{\mathrm{orb}}$.

This completes the proof of Theorem 5.3. □

Remark 5.4. As a by-product of the proof we have seen that the six invariants c_0, d_0, c_1, d_1, t_2, t_3 are sufficient for separating the points of $\mathcal{P}$. Hence, they define a homeomorphism of $\mathcal{P}$ onto the projection of $\tilde{\mathcal{P}}$ to $\mathbb{R}^6$. (Outside some 'momentum cut-off' $\|A\| \leq k$ the homeomorphism property is obvious and inside one uses that a bijection of a compact space onto a Hausdorff space is a homeomorphism.) The invariants c_2, d_2 cannot be expressed as polynomials in the other invariants, though. However, according to (34) and (35) and the subsequent discussion, on $\tilde{\mathcal{P}}$ they can be expressed as continuous functions in the other invariants. For $(c_0, d_0) \neq (3, 0), (-\frac{3}{2}, \pm\frac{3}{2}\sqrt{3})$,

$$c_2 = P_0^{-1}((3 - c_0)c_1^2 - (3 - c_0)d_1^2 - 2d_0c_1d_1 + 2(c_0(3 - c_0) + d_0^2)t_2), \tag{43}$$

$$d_2 = P_0^{-1}(d_0c_1^2 - d_0d_1^2 - 2(3 + c_0)c_1d_1 + 2d_0(3 + 2c_0)t_2), \tag{44}$$

whereas for $(c_0, d_0) = (3, 0), (-\frac{3}{2}, \pm\frac{3}{2}\sqrt{3})$, in the respective order,

$$(c_2, d_2) = (3t_2, 0), \left(-\frac{1}{2}t_2, \mp\frac{\sqrt{3}}{2}t_2\right). \tag{45}$$

One can extend c_2 and d_2 to rational functions on $\mathbb{R}^6$ by means of the expressions on the r.h.s. of (43) and (44). Then the values (45) have to be understood as the limits when $(c_0, d_0) \to (3, 0), (-\frac{3}{2}, \pm\frac{3}{2}\sqrt{3})$ along $\tilde{\mathcal{P}}$.

On the level of the semialgebraic sets, the projection $\tilde{\pi} : \tilde{\mathcal{P}} \to \tilde{\mathcal{X}}$ is just given by the natural projection to the c_0–d_0 plane. Fig. 5 shows the projections of the fibres $\tilde{\pi}^{-1}(c_0, d_0)$ to the three-dimensional subspace spanned by the coordinates t_2, t_3 and c_1 for five different points $(c_0, d_0) \in \tilde{\mathcal{X}}$. In addition, the projection of the fibres to the t_2–t_3 plane, which just coincides with $\tilde{\mathcal{Y}}$, is shown, too. The figures were drawn using a parametrization of the invariants, induced by a parametrization of the matrices a and A.

For (c_0, d_0) belonging to the stratum $\tilde{\mathcal{X}}_2$ the fibre $\tilde{\pi}^{-1}(c_0, d_0)$ is a full 2-plane, folded three times over the curve $P_2 = 0$ (Fig. 5(a) and (b)). The self-intersections present in these figures are remnants of the projection to the t_2–t_3–c_1 hyperplane. In fact, they correspond to solutions c_1 of (41) of multiplicity 2 where the factor $Q_1t_2 - 24c_1^2$ in (40) vanishes, so that both values of d_1 in (38) are allowed. Since the latter are distinct (unless $t_2 = 0$), the fictitious self-intersection of the fibre is not present in the full $\mathbb{R}^8$.

When (c_0, d_0) approaches the stratum $\tilde{\mathcal{X}}_1$, i.e., the curve $P_1 = 0$, the two halves of the plane come closer (Fig. 5(b)). For $(c_0, d_0) \in \tilde{\mathcal{X}}_1$ they meet each other and thus make the fibre a half-plane with a double fold (Fig. 5(c)). On moving (c_0, d_0) further along $\tilde{\mathcal{X}}_1$ towards one of the points of the stratum $\tilde{\mathcal{X}}_0$ the three layers of this half-plane approach each other (Fig. 5(d)) to finally merge to a 'sixth-plane' cone for $(c_0, d_0) \in \tilde{\mathcal{X}}_0$ (Fig. 5(e)). This illustrates the abstract description of the fibres in Section 3.4.

5.4. Stratification

We determine the equations and inequalities defining the strata of $\tilde{\mathcal{P}}$. We will make use of the discriminant of the characteristic polynomial χ_{aA}. Define

$$P_4(c_0(a, A), \dots, t_3(a, A)) := \mathrm{Re}(\mathrm{D}(\chi_{aA})).$$

Using (24) and (25) one finds

$$\chi_{aA}(z) = -z^3 + \mathrm{tr}(aA)z^2 + \left(\frac{1}{2}\mathrm{tr}(A^2)\overline{\mathrm{tr}(a)} - \overline{\mathrm{tr}(aA^2)}\right) z + \frac{1}{3}\mathrm{tr}(A^3).$$

It follows that

$$\begin{aligned} P_4 = {} & c_1^2 c_2^2 - c_1^2 d_2^2 + 4c_1 d_1 c_2 d_2 + d_1^2 d_2^2 - \frac{4}{3} t_3 c_1^3 - c_0 t_2 c_1^2 c_2 + d_0 t_2 c_1^2 d_2 + 4 t_3 c_1 d_1^2 \\ & - 2 d_0 t_2 c_1 d_1 c_2 - 2 c_0 t_2 c_1 d_1 d_2 + c_0 t_2 d_1^2 c_2 - d_0 t_2 d_1^2 d_2 - 4 c_2^3 \\ & + 12 c_2 d_2^2 + \frac{1}{4}(c_0^2 - d_0^2) t_2^2 c_1^2 + c_0 d_0 t_2^2 c_1 d_1 + 6 t_3 c_1 c_2 - \frac{1}{4}(c_0^2 - d_0^2) t_2^2 d_1^2 \\ & + 6 t_3 d_1 d_2 + 6 c_0 t_2 c_2^2 - 12 d_0 t_2 c_2 d_2 - 6 c_0 t_2 d_2^2 - 3 c_0 t_2 t_3 c_1 \\ & - 3 d_0 t_2 t_3 d_1 + 3(d_0^2 - c_0^2) t_2^2 c_2 + 6 c_0 d_0 t_2^2 d_2 + \frac{1}{2} c_0 (c_0^2 - 3 d_0^2) t_2^3 - 3 t_3^2. \end{aligned}$$

Theorem 5.5. *As subsets of $\tilde{\mathcal{P}}$, the strata $\tilde{\mathcal{P}}_k$ are defined by the following equations and inequalities:*

$$\begin{aligned} &\tilde{\mathcal{P}}_0 : P_0 = 0 \quad \textit{and} \quad t_2 = 0 \\ &\tilde{\mathcal{P}}_1 : P_1 = P_2 = P_4 = 0 \quad \textit{and} \quad (P_0 > 0 \text{ or } t_2 > 0) \\ &\tilde{\mathcal{P}}_2 : P_1 > 0 \quad \textit{or} \quad P_2 > 0 \quad \textit{or} \quad P_4 \neq 0 \end{aligned}$$

Proof. Let $(a, A) \in T \times \mathfrak{t}$ be given.

The pair (a, A) is invariant under the full S_3-action iff so are a and A individually. According to Theorem 5.1 and Lemma 5.2, this holds iff $P_0 = 0$ and $t_2 = 0$, respectively. Next, assume that (a, A) has nontrivial stabilizer. Then there are two entries which coincide for a and A simultaneously. Then aA has a degenerate eigenvalue. It follows that $\mathrm{D}(\chi_{aA}) = 0$ and, hence, $P_4 = 0$. Conversely, assume $P_1 = P_2 = P_4 = 0$. Then a and A both have coinciding entries. Up to S_3-action we can assume $a = \mathrm{diag}(\alpha, \alpha, \overline{\alpha}^2)$, $\alpha \in \mathrm{U}(1)$. Then A can be

$$\mathrm{diag}(\mathrm{i}x, \mathrm{i}x, -2\mathrm{i}x), \qquad \mathrm{diag}(\mathrm{i}x, -2\mathrm{i}x, \mathrm{i}x) \quad \text{or} \quad \mathrm{diag}(-2\mathrm{i}x, \mathrm{i}x, \mathrm{i}x), \quad x \in \mathbb{R}. \tag{46}$$

If $x = 0$ or $\alpha^3 = 1$ then in all three cases (a, A) has nontrivial stabilizer. Hence, assume $x \neq 0$ and $\alpha^3 \neq 1$. In the second and the third case,

$$\mathrm{D}(\chi_{aA}) = (\alpha x + 2\alpha x)^2 (\alpha x - \overline{\alpha}^2 x)^2 (2\alpha x + \overline{\alpha}^2 x)^2 = 9x^6 (2\alpha^3 - \overline{\alpha}^3 - 1)^2.$$

Taking the real part and replacing $\mathrm{Im}(\alpha^3)^2 = 1 - \mathrm{Re}(\alpha^3)^2$ yields

$$P_4 = 72x^6 (\mathrm{Re}(\alpha^3) - 1)^2 = 0.$$

Hence, $x = 0$ or $\alpha^3 = 1$, in contradiction to the assumption. Therefore, $A = \mathrm{diag}(\mathrm{i}x, \mathrm{i}x, -2\mathrm{i}x)$ and hence (a, A) has nontrivial stabilizer. This yields the equations for $\tilde{\mathcal{P}}_1$. The inequalities for $\tilde{\mathcal{P}}_1$ and $\tilde{\mathcal{P}}_2$ are obvious. □

5.5. Poisson structure

The brackets of the generating invariants $c_0, \ldots, t_3$, taken in the Poisson algebra $C^\infty(\mathcal{P})$, define a Poisson structure on $\mathbb{R}^8$ by

$$\{f, g\} := \sum_{i,j=1}^{8} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \{x_i, x_j\}, \tag{47}$$

where $(x_1, \ldots, x_8) = (c_0, \ldots, t_3)$. This Poisson structure rules the dynamics on $\tilde{\mathcal{P}}$; see the brief remark in Section 6. The Poisson brackets in $C^\infty(\mathcal{P})$ are defined by

$$\{f, g\} = \omega(X_f, X_g), \qquad f, g \in C^\infty(G \times \mathfrak{g}),$$

where the symplectic form ω is given by (9) and X_f, X_g are the Hamiltonian vector fields associated with f and g, respectively. They are defined pointwise by

$$\omega_{(a,A)}(X_f, X) = -X(f), \tag{48}$$

for all $X \in \mathrm{T}_{(a,A)}(G \times \mathfrak{g})$ and $(a, A) \in G \times \mathfrak{g}$. Here $X(f)$ is the directional derivative of f along X. As in Section 3.3 we write the tangent vectors in the form

$$(X_f)_{(a,A)} = (\mathrm{R}'_a B_f, (A, C_f)), \qquad X = (\mathrm{R}'_a B, (A, C))$$

with $B_f, C_f, B, C \in \mathfrak{g}$. Although it is not indicated by the notation, B_f and C_f depend on a and A, i.e., they are $\mathfrak{g}$-valued functions on $G \times \mathfrak{g}$. Using (9) and the invariance of the scalar product $\langle \cdot, \cdot \rangle$ to rewrite the l.h.s. of (48), and using the curve $(\exp(tB)a, A + tC)$ to represent X, (48) becomes

$$\langle B_f, C \rangle + \langle [B_f, A] - C_f, B \rangle = -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f((\exp(tB)a, A + tC)), \quad \forall\, B, C \in \mathfrak{g}. \tag{49}$$

Putting $B = 0$ yields B_f, then putting $C = 0$ and replacing B_f in the commutator yields C_f. Having found the Hamiltonian vector fields associated with the invariants this way, the Poisson brackets are then given pointwise by

$$\{f, g\}((a, A)) = \langle B_f, C_g \rangle - \langle C_f, B_g \rangle - \langle A, [B_f, B_g] \rangle. \tag{50}$$

Since it suffices to compute the brackets on the level set $J^{-1}(0)$, we may always assume $(a, A) \in J^{-1}(0)$. This simplifies the computations considerably. In particular, the commutators in (49) and (50) happen to vanish.

Let us illustrate the calculation by the bracket $\{c_1, d_1\}$. For c_1 and d_1, (49) reads

$$\langle B_{c_1}, C \rangle + \langle [B_{c_1}, A] - C_{c_1}, B \rangle = -\mathrm{Im}\langle a, C \rangle - \mathrm{Im}\langle aA, B \rangle$$
$$\langle B_{d_1}, C \rangle + \langle [B_{d_1}, A] - C_{d_1}, B \rangle = -\mathrm{Re}\langle a, C \rangle - \mathrm{Re}\langle aA, B \rangle.$$

To express the r.h.s. in terms of scalar products of B and C with elements of $\mathfrak{g}$, let Π_+ and Π_- denote the projections of $\mathrm{M}_3(\mathbb{C})$ onto the traceless Hermitian and traceless anti-Hermitian matrices, respectively. That is,

$$\Pi_\pm(D) = \frac{1}{2}(D \pm D^\dagger) - \frac{1}{6}\left(\mathrm{tr}(D) \pm \overline{\mathrm{tr}(D)}\right), \quad D \in \mathrm{M}_3(\mathbb{C}).$$

Both Π_- and $\mathrm{i}\Pi_+$ map $\mathrm{M}_3(\mathbb{C})$ to $\mathfrak{g}$ and for any $D \in \mathrm{M}_3(\mathbb{C})$ and $B \in \mathfrak{g}$ one has

$$\mathrm{Re}\langle D, B \rangle = \langle \Pi_-(D), B \rangle, \qquad \mathrm{Im}\langle D, B \rangle = \langle \mathrm{i}\Pi_+(D), B \rangle. \tag{51}$$

In this way, we obtain the Hamiltonian vector fields of the invariants:

$$
\begin{aligned}
&B_{c_0} = 0, && C_{c_0} = -\Pi_-(a) = -\frac{1}{2}(a-a^\dagger) + \frac{\mathrm{i}}{3}d_0,\\
&B_{d_0} = 0, && C_{d_0} = \mathrm{i}\Pi_+(a) = \frac{\mathrm{i}}{2}(a+a^\dagger) - \frac{\mathrm{i}}{3}c_0,\\
&B_{c_1} = -\mathrm{i}\Pi_+(a) = -\frac{\mathrm{i}}{2}(a+a^\dagger) + \frac{\mathrm{i}}{3}c_0, && C_{c_1} = \mathrm{i}\Pi_+(aA) = \frac{\mathrm{i}}{2}(a-a^\dagger)A + \frac{\mathrm{i}}{3}d_1,\\
&B_{d_1} = -\Pi_-(a) = -\frac{1}{2}(a-a^\dagger) + \frac{\mathrm{i}}{3}d_0, && C_{d_1} = \Pi_-(aA) = \frac{1}{2}(a+a^\dagger)A - \frac{\mathrm{i}}{3}c_1,\\
&B_{c_2} = -2\Pi_-(aA) = -(a+a^\dagger)A + \frac{2\mathrm{i}}{3}c_1, && C_{c_2} = \Pi_-(aA^2) = \frac{1}{2}(a-a^\dagger)A^2 + \frac{\mathrm{i}}{3}d_2,\\
&B_{d_2} = 2\mathrm{i}\Pi_+(aA) = \mathrm{i}(a-a^\dagger)A + \frac{2\mathrm{i}}{3}d_1, && C_{d_2} = -\mathrm{i}\Pi_+(aA^2) = -\frac{\mathrm{i}}{2}(a+a^\dagger)A^2 - \frac{\mathrm{i}}{3}c_2,\\
&B_{t_2} = -2A, && C_{t_2} = 0,\\
&B_{t_3} = 3\mathrm{i}\Pi_+(A^2) = 3\mathrm{i}A^2 + \mathrm{i}t_2, && C_{t_3} = 0.
\end{aligned}
$$

There hold the relations $B_{c_1} = -C_{d_0}$, $B_{d_1} = -C_{c_0}$, $B_{c_2} = -2C_{d_1}$, $B_{d_2} = 2C_{c_1}$. According to (50), e.g.,

$$
\begin{aligned}
\{c_1, d_1\} &= \langle B_{c_1}, C_{d_1}\rangle - \langle C_{c_1}, B_{d_1}\rangle = \langle -\mathrm{i}\Pi_+(a), C_{d_1}\rangle - \langle \mathrm{i}\Pi_+(aA), B_{d_1}\rangle\\
&= -\mathrm{Im}\langle a, C_{d_1}\rangle - \mathrm{Im}\langle aA, B_{d_1}\rangle.
\end{aligned}
$$

By replacing C_{d_1} and B_{c_1} using the above explicit expressions and rewriting the resulting scalar products in terms of the invariants $c_0, \ldots, t_3$ we finally arrive at the desired Poisson brackets:

$$
\begin{aligned}
&\{c_0, d_0\} = 0, && \{c_1, d_1\} = \frac{1}{3}(c_0c_1 + d_0d_1)\\
&\{t_2, t_3\} = 0, && \{c_2, d_2\} = -2t_3 + \frac{2}{3}(c_1c_2 + d_1d_2)\\
&\{c_0, c_1\} = -\frac{2}{3}c_0d_0 - d_0, && \{d_0, d_1\} = \frac{2}{3}c_0d_0 + d_0\\
&\{c_0, d_1\} = \frac{1}{2}c_0^2 - \frac{1}{6}d_0^2 - c_0 - \frac{3}{2}, && \{d_0, c_1\} = \frac{1}{6}c_0^2 - \frac{1}{2}d_0^2 - c_0 + \frac{3}{2}\\
&\{c_0, c_2\} = -c_0d_1 - \frac{1}{3}d_0c_1 + d_1, && \{d_0, d_2\} = \frac{1}{3}c_0d_1 + d_0c_1 - d_1\\
&\{c_0, d_2\} = c_0c_1 - \frac{1}{3}d_0d_1 + c_1, && \{d_0, c_2\} = \frac{1}{3}c_0c_1 - d_0d_1 + c_1
\end{aligned}
$$

$$
\begin{aligned}
\{c_1, c_2\} &= -\frac{5}{6}c_0d_2 - \frac{1}{2}d_0c_2 - \frac{1}{2}d_0t_2 + \frac{1}{2}d_2 + \frac{2}{3}c_1d_1\\
\{c_1, d_2\} &= \frac{5}{6}c_0c_2 - \frac{1}{2}d_0d_2 - \frac{1}{2}c_0t_2 - \frac{3}{2}t_2 + \frac{1}{2}c_2 + \frac{2}{3}d_1^2\\
\{d_1, c_2\} &= \frac{1}{2}c_0c_2 - \frac{5}{6}d_0d_2 - \frac{1}{2}c_0t_2 + \frac{1}{2}c_2 + \frac{3}{2}t_2 - \frac{2}{3}c_1^2\\
\{d_1, d_2\} &= \frac{1}{2}c_0d_2 + \frac{5}{6}d_0c_2 + \frac{1}{2}d_0t_2 - \frac{1}{2}d_2 - \frac{2}{3}c_1d_1
\end{aligned}
$$

$$\begin{aligned}
&\{c_0, t_2\} = -2d_1, && \{d_0, t_2\} = 2c_1 \\
&\{c_1, t_2\} = -2d_2, && \{d_1, t_2\} = 2c_2 \\
&\{c_2, t_2\} = -t_2 d_1 - \frac{2}{3} t_3 d_0, && \{d_2, t_2\} = t_2 c_1 + \frac{2}{3} t_3 c_0 \\
&\{c_0, t_3\} = t_2 d_0 - 3d_2, && \{d_0, t_3\} = -t_2 c_0 + 3c_2 \\
&\{c_1, t_3\} = -\frac{1}{2} t_2 d_1 - t_3 d_0, && \{d_1, t_3\} = \frac{1}{2} t_2 c_1 + t_3 c_0 \\
&\{c_2, t_3\} = -\frac{1}{2} t_2 d_2 - t_3 d_1, && \{d_2, t_3\} = \frac{1}{2} t_2 c_2 + t_3 c_1.
\end{aligned} \tag{52}$$

Remark 5.6. Another description of the reduced phase space in terms of invariants can be constructed as follows [7,8]. The polar map $(a, A) \mapsto a \exp(-\mathrm{i}A)$ yields a diffeomorphism of $T \times \mathfrak{t}$ onto the complexification $T^{\mathbb{C}}$, which is isomorphic to the direct product of two copies of the group of nonzero complex numbers. This diffeomorphism passes to an isomorphism of stratified symplectic space from $\mathcal{P}$ onto $T^{\mathbb{C}}/S_3$. The real invariants for the latter quotient are the elementary bisymmetric functions on $T^{\mathbb{C}}$, obtained from the elementary symmetric functions by bilinearization w.r.t. the holomorphic coordinates and their complex conjugates. This description is the starting point for stratified Kähler quantization in [6,7]. It also has the great advantage that it directly generalizes to $\mathrm{SU}(n)$ and further to an arbitrary compact Lie group. For classical dynamics, however, it has the drawback that the kinetic energy is not polynomial in the generating invariants.

6. Towards classical dynamics (an outlook)

In this final section, we make some general remarks on the dynamics on $\mathcal{P}$ and $\mathcal{X}$. A detailed study will be carried out in a subsequent paper.

In terms of the symplectic covering χ of Section 4, the dynamics can be described as follows. Given a Hamiltonian function $H \in C^\infty(\mathcal{P})$, the lift $\chi^* H$ is a Hamiltonian function on $\mathbb{R}^4$. Let the curve $(x(t), p(t))$ be a solution of the Hamiltonian equations associated with $\chi^* H$,

$$\dot{p}_j = -\frac{\partial(\chi^* H)}{\partial x^j}, \qquad \dot{x}^j = \frac{\partial(\chi^* H)}{\partial p_j}, \quad j = 1, 2. \tag{53}$$

To be a solution is a local property. Since the map $\psi : \mathbb{R}^4 \to T \times \mathfrak{t}$ is a local symplectomorphism, then $\psi((x(t), y(t)))$ is a solution of the Hamiltonian equations of $\lambda^* H$ on $T \times \mathfrak{t}$. According to point 2 of Remark 3.1, this curve stays inside $(T \times \mathfrak{t})_k$ for some $k = 2, 1, 0$. Hence, $(x(t), p(t))$ stays inside the corresponding $\mathbb{R}^4_k$ and $\chi((x(t), p(t))) = \chi_k((x(t), p(t)))$ is a curve in $\mathcal{P}_k$. Since χ_k is a local symplectomorphism by Theorem 4.2, then this curve is a solution of the Hamiltonian equations of the Hamiltonian function $H|_{\mathcal{P}_k}$ (restriction) on the stratum $\mathcal{P}_k$. This way, the Hamiltonian dynamics on $\mathcal{P}$ w.r.t. H is completely solved by the Hamiltonian dynamics w.r.t. $\chi^* H$ on $\mathbb{R}^4$. Furthermore, the trajectories in $\mathcal{X}$ are given by $\pi \circ \chi\ ((x(t), p(t)))$. Define $\tilde{\chi} : \mathbb{R}^2 \to \mathcal{X}$ to be the composition of the covering $\varphi : \mathbb{R}^2 \to T$, see (16), with the natural projection $T \to \mathcal{X}$. Then $\pi \circ \chi\ ((x(t), p(t))) = \tilde{\chi}(x(t))$. Hence, for the discussion of the trajectories in $\mathcal{X}$, it suffices to consider the trajectories $x(t)$ in $\mathbb{R}^2$. An explicit realization of the trajectories in $\mathcal{X}$ can be obtained by passing to $\mathbb{R}^2$ by means of the Hilbert map $\rho_{\mathcal{X}}$; see (26). One finds

$$\begin{aligned}
\rho_{\mathcal{X}} \circ \tilde{\chi}(x(t)) = \Bigg(&\cos\left(\frac{1}{\sqrt{6}} x^1(t) + \frac{1}{\sqrt{2}} x^2(t)\right) + \cos\left(\frac{1}{\sqrt{6}} x^1(t) - \frac{1}{\sqrt{2}} x^2(t)\right) + \cos\left(\sqrt{\frac{2}{3}} x^1(t)\right), \\
&\sin\left(\frac{1}{\sqrt{6}} x^1(t) + \frac{1}{\sqrt{2}} x^2(t)\right) + \sin\left(\frac{1}{\sqrt{6}} x^1(t) - \frac{1}{\sqrt{2}} x^2(t)\right) - \sin\left(\sqrt{\frac{2}{3}} x^1(t)\right)\Bigg).
\end{aligned}$$

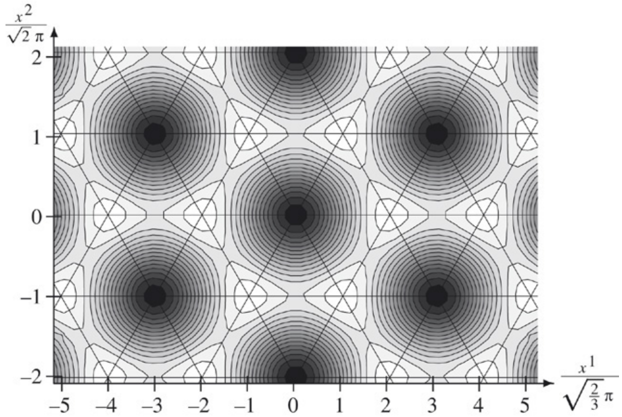

Fig. 6. Level diagram of the potential of the Hamiltonian (54). Dark regions mean low potential.

Now consider the Hamiltonian (2). One has

$$\chi^* H = \frac{\delta^3}{2}(p_1^2 + p_2^2) + \frac{1}{g^2\delta}\left(3 - \cos\left(\frac{x^1}{\sqrt{6}} + \frac{x^2}{\sqrt{2}}\right) - \cos\left(\frac{x^1}{\sqrt{6}} - \frac{x^2}{\sqrt{2}}\right) - \cos\left(\sqrt{\frac{2}{3}}x^1\right)\right). \tag{54}$$

The Hamiltonian has the standard structure, consisting of a kinetic energy term and a potential term. The potential is represented in Fig. 6. Its minimal value is 0; it is taken at the points

$$\left(3l\sqrt{\frac{2}{3}}\pi, (3l + 2m)\sqrt{2}\pi\right) \in \mathbb{R}^2_0, \quad l, m \in \mathbb{Z}.$$

The maximal value is $\frac{1}{g^2\delta}\frac{9}{2}$, taken at

$$\left((3l + 1)\frac{2}{\sqrt{3}}\pi, (3l + 2m + 1)\sqrt{2}\pi\right), \left((3l + 2)\frac{2}{\sqrt{3}}\pi, (3l + 2m + 2)\sqrt{2}\pi\right) \in \mathbb{R}^2_0, \quad l, m \in \mathbb{Z}.$$

In addition, the potential has saddle points at

$$\left(3l\sqrt{\frac{2}{3}}\pi, (3l + 2m + 1)\sqrt{2}\pi\right) \in \mathbb{R}^2_1, \quad l, m \in \mathbb{Z}.$$

In the representation of $\mathbb{R}^2$ in Fig. 3, the minima are the points labelled by 0; they project to $\mathbb{1} \in \mathcal{X}_0$. The maxima are the points labelled by 1 and 2; they project to the other two central elements $e^{i\frac{2}{3}\pi}\mathbb{1}$ and $e^{i\frac{4}{3}\pi}\mathbb{1} \in \mathcal{X}_0$. The saddle points are situated in the middle between points labelled 1 and 2.

The Hamiltonian equations associated with H are

$$\begin{aligned} \dot{p}_1 &= -\frac{1}{g^2\delta}\sqrt{\frac{2}{3}}\left(\sin\left(\frac{1}{\sqrt{6}}x^1\right)\cos\left(\frac{1}{\sqrt{2}}x^2\right) + \sin\left(\sqrt{\frac{2}{3}}x^1\right)\right), \\ \dot{p}_2 &= -\frac{1}{g^2\delta}\sqrt{2}\cos\left(\frac{1}{\sqrt{6}}x^1\right)\sin\left(\frac{1}{\sqrt{2}}x^2\right), \\ \dot{x}^j &= \delta^3 p_j, \quad j = 1, 2. \end{aligned} \tag{55}$$

Combining them, we obtain

$$\ddot{x}^1 + \frac{\delta^2}{g^2}\sqrt{\frac{2}{3}}\left(\sin\left(\frac{1}{\sqrt{6}}x^1\right)\cos\left(\frac{1}{\sqrt{2}}x^2\right) + \sin\left(\sqrt{\frac{2}{3}}x^1\right)\right) = 0,$$

$$\ddot{x}^2 + \frac{\delta^2}{g^2}\sqrt{2}\cos\left(\frac{1}{\sqrt{6}}x^1\right)\sin\left(\frac{1}{\sqrt{2}}x^2\right) = 0. \tag{56}$$

As mentioned above, this system of equations will be studied in detail in a subsequent paper.

Next, we comment on the discussion of the dynamics in terms of the invariants of Section 5. For a given Hamiltonian function $\tilde{H} \in C^\infty(\mathbb{R}^8)$, the dynamics takes place on $\mathbb{R}^8$ and is ruled by the Poisson structure defined by the brackets of the coordinates (52). That is, the equations of motion are given by

$$\dot{x}_j = \{\tilde{H}, x_j\}, \qquad (x_1, \ldots, x_8) = (c_0, \ldots, t_3). \tag{57}$$

By construction of the Poisson structure, $\tilde{\mathcal{P}}$ is invariant under the flow of $\tilde{H}$ for any $\tilde{H} \in C^\infty(\mathbb{R}^8)$. In terms of the invariants, the Hamiltonian (2) reads

$$\tilde{H} = \frac{\delta^3}{2}t_2 + \frac{1}{g^2\delta}(3 - c_0).$$

The second term corresponds to the potential term in (54). Its level lines in $\tilde{\mathcal{X}}$ are just straight lines parallel to the d_0-axis; cf. Fig. 4. The minimum is at the corner $(c_0, d_0) = (3, 0)$, the maxima are at the corners $(c_0, d_0) = (-\frac{3}{2}, \pm\sqrt{3}\frac{3}{2})$, the saddle point is at the boundary point $(c_0, d_0) = (-1, 0)$.

The corresponding equations of motion (57) yield a highly coupled system, which will not be reproduced here. At first sight it does not seem to be easier to handle than the equations of motion in terms of the symplectic covering (56). It will be a future task to study and unravel this system.

Acknowledgements

The authors would like to thank Sz. Charzynski, J. Huebschmann and I.P. Volobuev for helpful discussions on the invariants and the structure of the reduced phase space.

M.S. acknowledges funding by the German Research Council (DFG) under project nr. RU692/3.

References

[1] R. Abraham, J.E. Marsden, Foundations of Mechanics, Addison-Wesley, 1978.
[2] J.M. Arms, R.H. Cushman, M.J. Gotay, A universal reduction procedure for Hamiltonian group actions, in: The Geometry of Hamiltonian Systems, Berkeley, CA, 1989, in: Math. Sci. Res. Inst. Publ., vol. 22, Springer, New York, 1991, pp. 33–51.
[3] S. Charzyński, J. Kijowski, G. Rudolph, M. Schmidt, On the stratified classical configuration space of lattice QCD, J. Geom. Phys. 55 (2005) 137–178.
[4] S. Charzyński, G. Rudolph, M. Schmidt, On the topology of the reduced classical configuration space of lattice QCD, hep-th/0512129.
[5] R.H. Cushman, L.M. Bates, Global Aspects of Classical Integrable Systems, Birkhäuser-Verlag, Basel, 1997.
[6] J. Huebschmann, Kaehler spaces, nilpotent orbits, and singular reduction, Mem. Amer. Math. Soc. 172 (814) (2004);
J. Huebschmann, Lie–Rinehart, algebras, descent, and quantization, in: Galois Theory, Hopf Algebras, and Semiabelian Categories, in: Fields Inst. Commun., vol. 43, Amer. Math. Soc., Providence, RI, 2004, pp. 295–316.
[7] J. Huebschmann, Kähler quantization and reduction, J. Reine Angew. Math. 591 (2006). math.sg/0207166.
[8] J. Huebschmann, Classical phase space singularities and quantization, in: 4th International Symposium "Quantum Theory and Symmetries", Varna, 2005, in: V.K. Dobrev (Ed.), Quantum Theory and Symmetries IV, Heron Press, Sofia, 2006, pp. 51–68;
J. Huebschmann, Singular Poisson–Kähler geometry of certain adjoint quotients, in: Proceedings, The Mathematical Legacy of C. Ehresmann, Bedlewo, 2005, Banach centre publications (in press).
[9] D. Husemoller, Fibre Bundles, McGraw-Hill, 1966 (Springer 1994).
[10] P.D. Jarvis, J. Kijowski, G. Rudolph, On the structure of the observable algebra of QCD on the lattice, J. Phys. A 38 (2005) 5359–5377.
[11] J. Kijowski, G. Rudolph, Charge superselection sectors for qcd on the lattice, J. Math. Phys. 46 (2005) 032303.
[12] N.P. Landsman, M. Pflaum, M. Schlichenmaier (Eds.), Quantization of Singular Symplectic Quotients, in: Progr. Math., vol. 198, Birkhäuser, 2001.
[13] E. Lerman, R. Montgomery, R. Sjamaar, Examples of singular reduction, in: Symplectic Geometry, in: London Math. Soc. Lect. Note Ser., vol. 192, Cambridge University Press, Cambridge, 1993, pp. 127–155.
[14] J.P. Ortega, T. Ratiu, Momentum Maps and Hamiltonian Reduction, in: Progress Mathematics, vol. 222, Birkhäuser, 2004.
[15] M. Perlmutter, M. Rodriguez-Olmos, M.E. Sousa-Dias, On the geometry of reduced cotangent bundles at zero momentum, math.SG/0310437.
[16] M.J. Pflaum, Analytic and Geometric Study of Stratified Spaces, in: Lect. Notes Math., vol. 1768, Springer-Verlag, Berlin, 2001.
[17] C. Procesi, The invariant theory of $n \times n$ matrices, Adv. Math. 19 (1976) 306–381.
[18] G.W. Schwarz, Smooth functions invariant under the action of a compact Lie group, Topology 14 (1975) 63–68.
[19] R. Sjamaar, E. Lerman, Stratified symplectic spaces and reduction, Ann. of Math. 134 (2) (1991) 375–422.
[20] H. Weyl, The Classical Groups, Princeton University Press, 1966, Section VIII.14.